This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

EducT 148.96.874

Harvard College Library

FROM

Miss Ellen L. Wentworth
Exeter, N.H.

3 2044 097 044 051

EXERCISES

IN

WENTWORTH'S GEOMETRY,

REVISED EDITION,

WITH SOLUTIONS.

Boston, U.S.A., and London:
GINN & COMPANY, PUBLISHERS.
1896.

Educ T 148.96.874

HARVARD COLLEGE
Mar. 29, 1939
LIBRARY

Miss Ellen L. Wentworth
Exeter, N. H.

Entered according to Act of Congress, in the year 1888, by
G. A. WENTWORTH,
in the Office of the Librarian of Congress at Washington.

Typography by J. S. Cushing & Co., Boston, U.S.A.

Presswork by Ginn & Co., Boston, U.S.A.

PREFACE.

THIS book is intended for teachers, *and for them only*. The publishers will sell it to no persons except teachers of Wentworth's Geometry. Every teacher must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn & Company.

It is hoped that young teachers will derive great help from studying the systematic solutions of the exercises, and that every teacher who is pressed for time will find great relief by not being obliged to work out every problem himself.

G. A. WENTWORTH.

EXETER, N. H.,
1888.

TEACHERS' EDITION.

Book I.

Ex. 1. Find the value of an angle if it is double its complement; if it is one-fourth its complement.

(1) Let $x =$ the angle; then $x + \frac{x}{2} = 90°$; whence $x = 60°$.

(2) Let $x =$ the angle; then $x + 4x = 90°$; whence $x = 18°$.

Ex. 2. Find the value of an angle if it is double its supplement; if it is one-third its supplement.

(1) Let $x =$ the angle; then $x + \frac{x}{2} = 180°$; whence $x = 120°$.

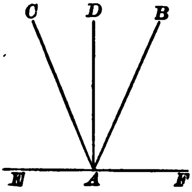
(2) Let $x =$ the angle; then $x + 3x = 180°$; whence $x = 45°$.

Ex. 3. How many degrees in the angle formed by the hands of a clock at 2 o'clock? 3 o'clock? 4 o'clock? 6 o'clock?

Ans. 60°, 90°, 120°, 180°.

Ex. 4. If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.

Let BAC be the given angle, AD the bisector. Through A draw $EF \perp$ to AD; then $\angle CAE = \angle BAF$.



Proof. $\angle CAE = 90° - \angle DAC$,

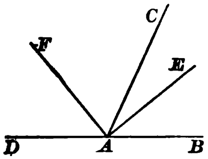
$\angle BAF = 90° - \angle DAB$.

But $\angle DAC = \angle DAB$.

$\therefore \angle CAE = \angle BAF$.

Ex. 5. If the bisectors of two adjacent angles are perpendicular to each other, the adjacent angles are supplementary.

Let $\measuredangle$ BAC, CAD, be two adj. $\measuredangle$, AE bisect $\angle BAC$, AF bisect $\angle CAD$, and $\angle EAF = 90°$.



Then $\angle BAC + \angle CAD = 180°$.

PROOF.

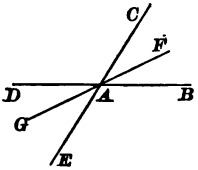
Since $\angle EAC + \angle CAF = 90°$,

$\therefore 2\angle EAC + 2\angle CAF = 180°$;

or $\angle BAC + \angle CAD = 180°$.

Ex. 6. The bisector of one of two vertical angles bisects the other.

Let $\measuredangle$ BAC, DAE, be two vertical $\measuredangle$. Let AF bisect $\angle BAC$. Produce FA to G.



Then $\angle DAG = \angle EAG$.

PROOF. $\angle DAG = \angle BAF$, (§ 95)

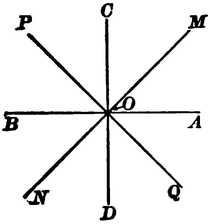
$\angle EAG = \angle CAF$.

But $\angle BAF = \angle CAF$.

$\therefore \angle DAG = \angle EAG$. (Ax. 1)

Ex. 7. The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.

Let AB and CD intersect at O, and let MN and PQ be the bisectors of the $\measuredangle$.



Then MN is $\perp$ to PQ.

PROOF. $\angle AOM = \angle MOC$,

and $\angle AOQ = \angle BOP = \angle COP$.

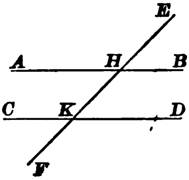
$\therefore \angle MOQ = \angle MOP$.

But $\angle MOQ + \angle MOP = 180°$.

$\therefore \angle MOQ = 90°$.

$\therefore$ MN is $\perp$ to PQ.

Ex. 8. If the angle AHE is an angle of 135°, find the number of degrees in each of the other angles formed at the points H and K.



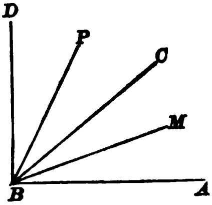
Ans. $\angle BHE = 45°$, $\angle AHK = 45°$,

$\angle KHB = 135°$, $\angle CKH = 135°$,

$\angle HKD = 45°$, $\angle CKF = 45°$,

$\angle FKD = 135°$.

Ex. 9. Find the angle between the bisectors of adjacent complementary angles.



Let $\angle$s ABC, CBD, be the adj. complementary $\angle$s. Let MB bisect $\angle ABC$, PB bisect $\angle CBD$.

To find the $\angle MBP$. By the given conditions,

$$2\angle MBC + 2\angle CBP = 90°.$$

$$\therefore \angle MBC + \angle CBP = 45°.$$

But $$\angle MBC + \angle CBP = \angle MBP.$$

$$\therefore \angle MBP = 45°.$$

Ex. 10. It has been shown that if two parallels are cut by a transversal, the alternate-interior angles are equal, the exterior-interior angles are equal, the two interior angles on the same side of the transversal are supplementary. State the opposite theorems. State the converse theorems.

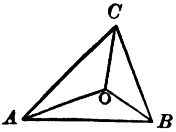
The opposite theorems are as follows:

If two lines not parallel are cut by a transversal, the alternate-interior angles are not equal, the exterior-interior angles are not equal, the two interior angles on the same side of the transversal are not supplementary.

The converse theorems are stated on pp. 25, 27, 29, of the Geometry.

That is, BD and BC are unequal.

Ex. 19. The sum of the lines which join a point within a triangle to the three vertices is less than the perimeter, but greater than half the perimeter.



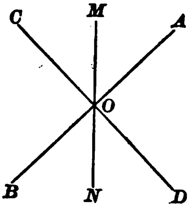
Let O be a point within $\triangle ABC$.

Then $$OA + OB + OC < AB + BC + CA,$$

and $$OA + OB + OC > \tfrac{1}{2}(AB + BC + CA).$$

PROOF $OA + OB < AC + BC$ (§ 118), but $> AB$ (§ 33)

Ex. 12. Show that the bisectors of two vertical angles form one straight line.



Let AB cut CD at O, forming the vertical $\angle$s AOC, BOD. Let OM bisect $\angle AOC$, and ON bisect $\angle BOD$. Then MON is a straight line.

PROOF. $\angle AOC = \angle BOD$. (§ 95)

$\therefore \angle AOM = \angle BON$. (Ax. 7)

Now $\angle AOM + \angle MOB = 180°$. (§ 90)

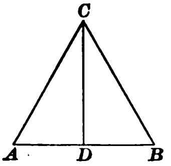
$\therefore \angle BON + \angle MOB = 180°$.

$\therefore MON$ is a straight line. (§ 93)

Ex. 13. Find the complement, and the supplement, of an angle containing 26° 52′ 37″.

Complement = 63° 7′ 23″; supplement = 153° 7′ 23″.

Ex. 14. The bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.



In $\triangle ABC$, let $AC = BC$, and CD bisect $\angle C$.

Then $AD = BD$, and CD is $\perp$ to AB.

Proof. In $\triangle ADC$, BDC,

$AC = BC$, (Hyp.)

$\angle ACD = \angle BCD$, (Hyp.)

and CD is common.

$\therefore \triangle ADC = \triangle BDC$. (§ 150)

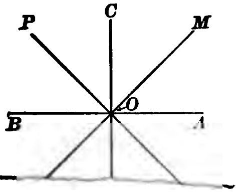
$\therefore AD = BD$, and $\angle ADC = \angle BDC = 90°$. (§ 45)

$\therefore CD$ is $\perp$ to AB.

Ex. 15. The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.

In $\triangle ABC$, let $AC = BC$, and let DE be drawn $\perp$ to AB through D,

Ex. 7. The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.



Let AB and CD intersect at O, and let MN and PQ be the bisectors of the $\angle$s.

Then MN is $\perp$ to PQ.

Proof. $\angle AOM = \angle MOC$,

and $\angle AOQ = \angle BOP = \angle COP$.

$\therefore \angle MOQ = \angle MOP$.

But $\angle MOQ + \angle MOP = 180°$.

Ex. 16. The perpendicular from the vertex to the base of an isosceles triangle bisects the base and the angle at the vertex.

In $\triangle ABC$, let $AC = BC$, and let CD be drawn through $C \perp$ to AB, meeting AB at D.

Then $AD = BD$, and $\angle ACD = \angle BCD$.

Proof. In rt. $\triangle ACD$, BCD,

$AC = BC$, and $\angle CAD = \angle CBD$. (§ 154)

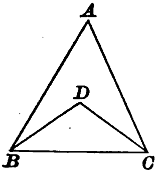
$\therefore \triangle ACD = \triangle BCD$. (§ 148)

$\therefore AD = BD$, and $\angle ACD = \angle BCD$.

Ex. 17. If the angles ABC and ACB, at the base of an isosceles triangle, be bisected by the straight lines BD, CD, show that DBC will be an isosceles triangle.

In $\triangle ABC$, let $AB = AC$, BD bisect $\angle ABC$, and CD bisect $\angle ACB$.

Then $\triangle DBC$ is isosceles.



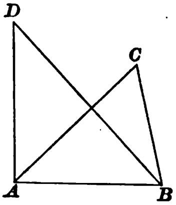
Proof. $\angle ABC = \angle ACB$. (§ 154)

$\therefore \angle DBC = \angle DCB$. (Ax. 7)

$\therefore CD = BD$,

and $\triangle DBC$ is isosceles. (§ 156)

Ex. 18. ABC and ABD are two triangles on the same base AB, and on the same side of it, the vertex of each triangle being without the other. If AC equal AD, show that BC cannot equal BD.

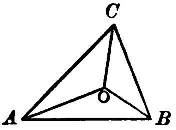


Proof. Since AC and AD do not coincide, one of these lines must make with AB an angle greater than the other.

Suppose $\angle DAB$ greater than $\angle CAB$; then BD is greater than BC. (§ 152)

That is, BD and BC are unequal.

Ex. 19. The sum of the lines which join a point within a triangle to the three vertices is less than the perimeter, but greater than half the perimeter.



Let O be a point within $\triangle ABC$.

Then $OA + OB + OC < AB + BC + CA$,

and $OA + OB + OC > \frac{1}{2}(AB + BC + CA)$.

Proof. $OA + OB < AC + BC$ (§ 118), but $> AB$ (§ 33)

$OB + OC < BA + CA$, but $> BC$

$OC + OA < AB + BC$, but $> AC$

$\therefore 2(OA + OB + OC) < 2(AB + BC + AC)$, but $> AB + BC + AC$,

or $OA + OB + OC < AB + BC + AC$, but $> \frac{1}{2}(AB + BC + AC)$.

Ex. 20. What is the locus of a point at a given distance from a fixed point?

Ans. The circumference of a circle having the fixed point for centre and the given distance for radius.

Ex. 21. What is the locus of a point equidistant from two fixed points?

Ans. The perpendicular bisector of the line joining the two points.

Ex. 22. What is the locus of a point at a given distance from a fixed straight line of indefinite length?

Ans. The two parallels drawn at the given distance from the fixed straight line.

Ex. 23. What is the locus of a point equidistant from two given parallel lines?

Ans. The straight line parallel to the given lines and drawn equidistant from them.

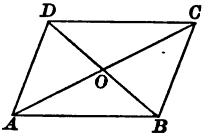
Ex. 24. What is the locus of a point equidistant from the extremities of a given line?

Ans. The perpendicular which bisects the line.

Ex. 25. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Let the diagonals AC, BD of the quadrilateral $ABCD$ meet at O, and let $AO = CO$, $BO = DO$.

Then the figure $ABCD$ is a parallelogram.



Proof. In $\triangle$s ABO, DCO,

$AO = CO$, (Hyp.)

$BO = DO$, (Hyp.)

$\angle AOB = \angle COD$. (§ 95)

$\therefore \triangle ABO = \triangle DCO$. (§ 150)

$\therefore AB = DC$, and $\angle BAO = \angle DCO$.

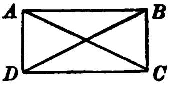
$\therefore AB$ is $\parallel$ to DC. (§ 105)

$\therefore$ figure $ABCD$ is a parallelogram. (§ 182)

Ex. 26. The diagonals of a rectangle are equal.

Let $ABCD$ be a rectangle. Then $AC = BD$.

Proof. In $\triangle$s ABD, DCA, AD is common.



$AB = DC$, (§ 179)

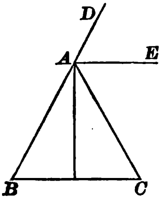
and $\angle BAD = \angle CDA$. (§ 169)

$\therefore \triangle ABD = \triangle DCA$, (§ 150)

and $AC = BD$.

In $\triangle ABC$, let AE bisect $\angle CAD$ formed by producing BA to any point D, and let AE be $\parallel$ to BC.

Then $AB = AC$, or $\triangle ABC$ is isosceles.



Proof. $\angle DAE = \angle ABC$, (§ 106)

and $\angle CAE = \angle ACB$. (§ 104)

Now $\angle DAE = \angle CAE$. (Hyp.)

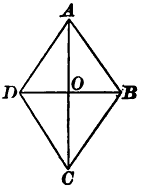
$\therefore \angle ABC = \angle ACB$.

$\therefore$ $AB = AC$, and $\triangle ABC$ is isosceles.

Ex. 38. The altitudes upon the legs of an isosceles triangle are equal.

In $\triangle ABC$, let $AB = AC$, and let BD be $\perp$ to AC, $CE \perp$ to AB. Then $BD = CE$.

Let $ABCD$ be a rhombus, and let AC meet BD at O. Then AC is $\perp$ to BD, and $\angle BAO = \angle DAO$, etc.



Proof. In $\triangle\triangle$ OAB, OAD, the side OA is common.

$AB = AD$, (§ 172)

and $OB = OD$. (§ 184)

$\therefore \triangle OAB = \triangle OAD$, (§ 160)

and $\angle BAO = \angle DAO$.

Also, $\angle AOB = \angle AOD$,

and $\therefore \angle AOB = 90°$. (§ 45)

$\therefore AC$ is $\perp$ to BD. Similarly, we may prove that $ABO = CBO$, etc.

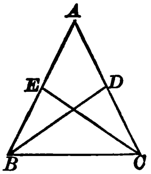
$\therefore \triangle BCD = \triangle BCE$. (§ 161) $\therefore \angle ACB = \angle ABC$.

$\therefore$ $AB = AC$, (§ 156), and $\triangle$ is isosceles.

Ex. 40. The medians drawn to the legs of an isosceles triangle are equal.

In $\triangle ABC$, let $AB = AC$, D be the middle point of AC, E the middle point of AB. Join BD, CE.

Then $BD = CE$.



Proof. In $\triangle\triangle$ BCD, BCE, BC is common,

$CD = BE$, (Ax. 7)

$\angle ACB = \angle ABC$. (§ 154)

$\therefore \triangle BCD = \triangle BCE$, (§ 150)

and $BD = CE$.

Ex. 21. What is the locus of a point equidistant from two fixed points?

Ans. The perpendicular bisector of the line joining the two points.

Ex. 22. What is the locus of a point at a given distance from a fixed straight line of indefinite length?

Ans. The two parallels drawn at the given distance from the fixed straight line.

Ex. 23. What is the locus of a point equidistant from two given parallel lines?

Ans. The straight line parallel to the given lines and drawn equidistant from them.

Ex. 35. ~~State and prove the conver~~se.

If the median from the vertex to the base of a triangle is perpendicular to the base, the triangle is isosceles.

In the $\triangle ABC$, let D be the middle point of BC, and let AD be $\perp$ to BC. Then $AB = AC$, or $\triangle ABC$ is isosceles.

Proof. In $\triangle$s BAD, CAD,

AD is common, $BD = CD$, and $\angle ADB = \angle ADC = 90°$. (Hyp.)

$\therefore \triangle BAD = \triangle CAD$. (§ 150)

$\therefore AB = AC$.

Ex. 36. The bisector of an exterior angle of an isosceles triangle, formed by producing one of the legs through the vertex, is parallel to the bas

$\angle AOB = \angle COD$. (§ 95)

$\therefore \triangle ABO = \triangle DCO$. (§ 150)

$\therefore AB = DC$, and $\angle BAO = \angle DCO$.

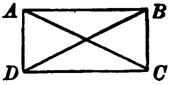
$\therefore AB$ is $\parallel$ to DC. (§ 105)

$\therefore$ figure $ABCD$ is a parallelogram. (§ 182)

Ex. 26. The diagonals of a rectangle are equal.

Let $ABCD$ be a rectangle. Then $AC = BD$.

Proof. In $\triangle$s ABD, DCA, AD is common.



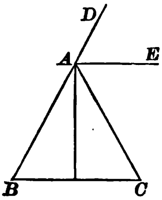
$AB = DC$, (§ 179)

and $\angle BAD = \angle CDA$. (§ 169)

$\therefore \triangle ABD = \triangle DCA$, (§ 150)

and $AC = BD$.

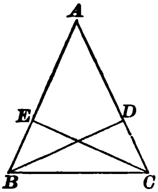
In $\triangle ABC$, let AE bisect $\angle CAD$ formed by producing BA to any point D, and let AE be $\parallel$ to BC.



Then $AB = AC$, or $\triangle ABC$ is isosceles.

Proof. $\angle DAE = \angle ABC$, (§ 106)
and $\angle CAE = \angle ACB$. (§ 104)
Now $\angle DAE = \angle CAE$. (Hyp.)
$\therefore \angle ABC = \angle ACB$.
$\therefore$ $AB = AC$, and $\triangle ABC$ is isosceles.

Ex. 38. The altitudes upon the legs of an isosceles triangle are equal.



In $\triangle ABC$, let $AB = AC$, and let BD be $\perp$ to AC, $CE \perp$ to AB. Then $BD = CE$.

Proof. In rt. $\triangle$s BCE, BCD, BC is common,
and $\angle CBE = \angle BCD$. (§ 154)
$\therefore \triangle BCE = \triangle BCD$. (§ 149)
$\therefore$ $BD = CE$.

Ex. 39. State and prove the converse.

If the altitudes upon two sides of a triangle are equal, the triangle is isosceles.

In $\triangle ABC$, let BD be $\perp$ to AC, $CE \perp$ to AB, and $BD = CE$.

Then $AB = AC$, and $\triangle ABC$ is isosceles.

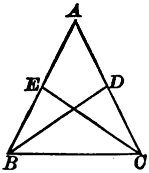
Proof. In rt. $\triangle$s BCD, BCE, BC is common, and $BD = CE$. (Hyp.)
$\therefore \triangle BCD = \triangle BCE$. (§ 161) $\therefore \angle ACB = \angle ABC$.
$\therefore$ $AB = AC$, (§ 156), and $\triangle$ is isosceles.

Ex. 40. The medians drawn to the legs of an isosceles triangle are equal.

In $\triangle ABC$, let $AB = AC$, D be the middle point of AC, E the middle point of AB. Join BD, CE.

Then $BD = CE$.

Proof. In $\triangle$s BCD, BCE, BC is common,
$CD = BE$, (Ax. 7)
$\angle ACB = \angle ABC$. (§ 154)
$\therefore \triangle BCD = \triangle BCE$, (§ 150)
and $BD = CE$.



of a triangle

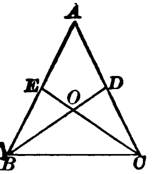
Ex. 41. A triangle is isosceles if the medians to two sides are equal.

In $\triangle ABC$, let BD, CE be two medians meeting at O, and $BD = CE$. Then $AB = AC$, and $\triangle ABC$ is isosceles.

Proof. In ⩓ BOE and COD

$BO = CO$, and $OE = OD$. (Ex. 33)
$\angle BOE = \angle COD$. (§ 95)
$\therefore \triangle BOE = \triangle COD$. (§ 150)
$\therefore BE = CD$.
$\therefore AB = AC$.

Pb a Δ is iso

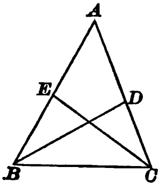


Ex. 42. The bisectors of the base angles of an isosceles triangle are equal.

In $\triangle ABC$, let $AB = AC$, and let BD bisect the $\angle ABC$, and CE the $\angle ACB$. Then $BD = CE$.

Proof. In ⩓ BCD, BCE, BC is common,

$\angle BCD = \angle ABC$, (§ 154)
and $\angle CBD = \angle BCE$. (Ax. 7)
$\therefore \triangle BCD = \triangle BCE$, (§ 147)
and $BD = CE$.



Ex. 43. State ~~the converse and~~ the opposite theorems.

Converse Theorem. If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

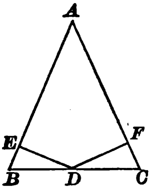
Opposite Theorem. The bisectors of two angles of a scalene triangle are unequal.

Ex. 44. The perpendiculars dropped from the middle point of the base of an isosceles triangle upon the legs are equal.

In $\triangle ABC$, let $AB = AC$, D be the middle point of BC, $DE \perp$ to AB, $DF \perp$ to AC. Then $DE = DF$.

Proof. In rt. ⩓ BDE, CDF,

$BD = CD$, (Hyp.)
$\angle EBD = \angle FCD$. (§ 154)
$\therefore \triangle BDE = \triangle CDF$, (§ 149)
and $DE = DF$.



Ex. 45. State and prove the converse.

If the perpendiculars dropped from the middle point of one side of a triangle to the other two sides are equal, the triangle is isosceles.

Let ABC be any $\triangle$, D the middle point of BC, $DE \perp$ to AB, $DF \perp$ to AC, and $DE = DF$.

Then $AB = AC$, or $\triangle ABC$ is isosceles.

PROOF. In rt. $\triangle$s BDE, CDF,

$$BD = DC, \text{ and } DE = DF. \qquad \text{(Hyp.)}$$

$$\therefore \triangle BDE = \triangle CDF \text{ (§ 161), and } \angle ABC = \angle ACB.$$

$$\therefore \quad AB = AC \text{ (§ 156), or } \triangle ABC \text{ is isosceles.}$$

Ex. 46. If one of the legs of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the leg produced to the nearer end of the base is perpendicular to the base.

In $\triangle ABC$, let $AC = BC$; produce AC, making $CD = AC$, and join DB. Then BD is $\perp$ to AB.

PROOF. In $\triangle BCD$, $CD = CB$ (since each $= AC$)

$$\therefore \angle CBD = \angle CDB; \qquad \text{(§ 154)}$$

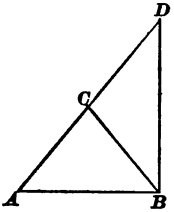
also $\angle CBA = \angle CAB$.

$\therefore$ by addition,

$$\angle ABD = \angle CDB + \angle CAB.$$

But sum of these $\angle$s $= 180°$. (§ 138)

$$\therefore \angle ABD = 90°. \quad \therefore BD \text{ is } \perp \text{ to } AB.$$



Ex. 47. Show that the sum of the interior angles of a hexagon is equal to eight right angles.

Let x denote the sum; then, by § 205,

$$x = (6-2) \times 2 \text{ rt. } \angle\text{s} = 4 \times 2 \text{ rt. } \angle\text{s} = 8 \text{ rt. } \angle\text{s}.$$

Ex. 48. Show that each angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

$$\text{Sum of } \angle\text{s of a pentagon} = 3 \times 2 \text{ rt. } \angle\text{s} = 6 \text{ rt. } \angle\text{s}. \qquad \text{(§ 205)}$$

$$\therefore \text{each angle} = \tfrac{1}{5} \text{ of } 6 \text{ rt. } \angle\text{s} = \tfrac{6}{5} \text{ of a rt. } \angle.$$

Ex. 49. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

Let x denote the number of sides; then, by § 205,

$$\text{Sum of the } \angle\text{s} = (x-2)180°.$$

$$\therefore \text{each } \angle = \frac{(x-2)180°}{x}. \quad \therefore \frac{4(x-2)180°}{x} = 630°.$$

$$720x - 1440 = 630x; \; 90x = 1440; \; x = 16.$$

Ex. 50. How many sides has a polygon, the sum of whose interior angles is equal to the sum of its exterior angles?

Let x denote the number of sides.
Then sum of interior $\angle$s $= (x - 2)\,180°$. (§ 205)
And sum of exterior $\angle$s $= 360°$. (§ 207)
$\therefore (x - 2)\,180° = 360°$; whence $x = 4$.

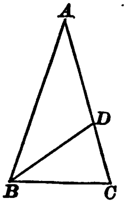
Ex. 51. How many sides has a polygon, the sum of whose interior angles is double that of its exterior angles?

Let x denote the number of sides.
Then proceeding as in the last exercise, we have the equation,
$(x - 2)\,180° = 720°$; whence $x = 6$.

Ex. 52. How many sides has a polygon, the sum of whose exterior angles is double that of its interior angles?

Let x denote the number of sides.
Then $360° = 2(x - 2)\,180°$; whence $x = 3$.

Ex. 53. BAC is a triangle having the angle B double the angle A. If BD bisect the angle B, and meet AC in D, show that BD is equal to AD.



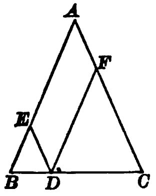
In $\triangle ABC$, let $\angle B = 2\angle A$, and BD bisect $\angle B$.

Then $BD = AD$.

PROOF. $\angle A = \frac{1}{2}\angle ABC = \angle ABD$. (Hyp.)
$\therefore BD = AD$. (§ 156)

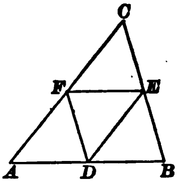
Ex. 54. If from any point in the base of an isosceles triangle parallels to the legs are drawn, show that a parallelogram is formed whose perimeter is constant, and equal to the sum of the legs of the triangle.

In $\triangle ABC$, let $AB = AC$, let D be any point in BC, DE be $\parallel$ to CA, $DF \parallel$ to BA. Then $AEDF$ is a ▱, and its perimeter is equal to $AB + AC$.



PROOF. $AEDF$ is a ▱ by definition. (§ 168)
$\angle EDB = \angle C$, (§ 106)
and $\angle B = \angle C$. (§ 154)
$\therefore \angle EDB = \angle B$.
$\therefore ED = EB$. (§ 156)
Similarly it can be proved that $FD = FC$.
$\therefore AE + ED + DF + AF = AB + AC$.

Ex. 55. The lines joining the middle points of the sides ~~of a triangle~~ divide the ~~triangle~~ into four equal triangles.



In $\triangle ABC$ let D, E, F be the middle points of the sides AB, BC, CA, respectively.

Then $\triangle$s ADF, DFE, DEB, CEF are equal.

Proof. In $\triangle$s ADF, DEF, DF is common,

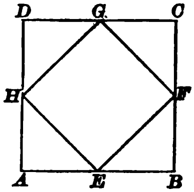
$AD = FE$, and $AF = DE$. (§ 189)

$\therefore \triangle ADF = \triangle DEF$. (§ 160)

Similarly, $\triangle DEB = \triangle DFE = \triangle CEF$.

Ex. 56. The lines joining the middle points of the sides of a square, taken in order, enclose a square.

Let $ABCD$ be a square, and E, F, G, H the middle points of its sides, taken in order. Then the figure $EFGH$ is also a square.



Proof. The rt. $\triangle$s AEH, BEF, CFG, DGH, are isosceles because their legs are each half of a side of the square.

And these four $\triangle$s are also equal. (§ 151)

$\therefore EF = FG = GH = HE$.

Also in each $\triangle$ the $\angle$s at the base are equal (§ 154).

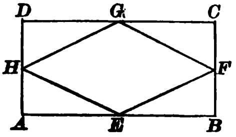
$\therefore$ each of the base $\angle$s $= 45°$. (§ 143)

$\therefore \angle FEH = 180° - \angle AEH - \angle BEF = 90°$.

Similarly, the other $\angle$s of $EFGH$ are rt. $\angle$s.

$\therefore$ the figure $EFGH$ is a square. (§ 171)

Ex. 57. The lines joining the middle points of the sides of a rectangle, taken in order, enclose a rhombus.



Let $ABCD$ be a rectangle, and E, F, G, H be the middle points of its sides, taken in order. Then the figure $EFGH$ is a rhombus.

Proof. In rt. $\triangle$s AEH, BEF,

$AE = BE$, (Hyp.)

and $AH = BF$. (§ 179)

$\therefore \triangle AEH = \triangle BEF$, (§ 151)

and $EH = EF$.

Similarly, we may prove that $EH = HG = GF$.

$\therefore EFGH$ is a square or a rhombus. (§§ 171, 172)

Now the figure *EFGH* cannot be a square. If it were a square, then

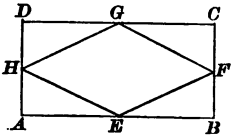
$\angle HEF = 90^\circ$.

$\therefore \angle AEH + \angle BEF = 90^\circ$. (§ 92)

But $\angle AEH = \angle BEF$. (equal $\triangle$s)

$\therefore \angle AEH = 45^\circ$. $\therefore \angle AHE = 45^\circ$. (§ 143)

$\therefore AE = AH$. (§ 156)



But this conclusion contradicts the hypothesis that the figure *ABCD* is any rectangle.

$\therefore$ the figure *EFGH* must be a rhombus.

Ex. 58. The lines joining the middle points of the sides of a rhombus, taken in order, enclose a rectangle.

Let *ABCD* be a rhombus, *E*, *F*, *G*, *H* the middle points of its sides, taken in order.

Then the figure *EFGH* is a rectangle.

Proof. Draw the diagonals *AC*, *BD*, meeting at *O*.

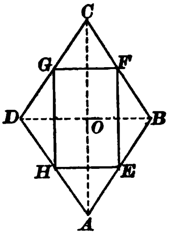
EF is ∥ to *AC* and equal to $\frac{1}{2}AC$ (§ 189).

HG is ∥ to *AC* and equal to $\frac{1}{2}AC$.

$\therefore$ *EF* is ∥ to *HG* (§ 111), and $EF = HG$ (Ax. 1). $\therefore$ figure *EFGH* is a ▱ (§ 182).

The $\angle$s formed at *O* are rt. $\angle$s (Ex. 28).

$\therefore$ the $\angle$s of the ▱ *EFGH* are rt. $\angle$s (§ 112).



Also *BD* is not equal to *AC*. $\therefore$ *EF* is not equal to *FG*.

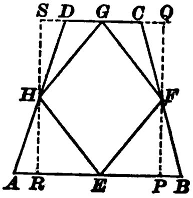
$\therefore$ the figure *EFGH* must be a rectangle.

Ex. 59. The lines joining the middle points of the sides of an isosceles trapezoid, taken in order, enclose a rhombus or a square.

Let *ABCD* be an isosceles trapezoid, and let *EF*, *FG*, *GH*, *HE* join the middle points of its sides.

Then the figure *EFGH* is a rhombus or a square.

Proof. Through the middle points *F*, *H* of the legs draw *PQ*, *RS* ⊥ to the bases, *PQ* meeting *AB* at *P* and *DC* produced at *Q*, *RS* meeting *AB* at *R* and *CD* produced at *S*. Then *PQ* is ∥ to *RS* (§ 100).



Also *RP* is ∥ to *SQ* (hyp.). $\therefore$ figure *PQSR* is a ▱ (§ 168).

And, by construction, *PQSR* is a rectangle or a square.

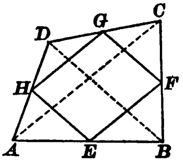
$\therefore$ the figure *EFGH* is a rhombus (Ex. 57) or a square (Ex. 56).

Put in after 55

Ex. 60. The lines joining the middle points of the sides of any quadrilateral, taken in order, enclose a parallelogram.

Let $ABCD$ be a quadrilateral, E, F, G, H the middle points of the sides, taken in order.

Then the figure $EFGH$ is a ▱.



PROOF. Draw the diagonals AC, BD.

EF and HG are $\parallel$ to AC. (§ 189)

$\therefore$ EF is $\parallel$ to HG. (§ 111)

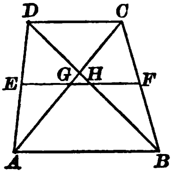
Similarly, EH is proved $\parallel$ to FG.

$\therefore$ figure $EFGH$ is a ▱. (§ 168)

Ex. 61. The median of a trapezoid passes through the middle points of the two diagonals.

Let $ABCD$ be a trapezoid, AB, DC the $\parallel$ sides, AC, BD the diagonals, EF the median.

Then EF bisects AC and BD.



PROOF. Let G, H be the middle points of AC, BD.

EF is $\parallel$ to AB and CD. (§ 191)

E is the middle point of AD. (Hyp.)

$\therefore$ EF passes through H. (§ 188)

For a similar reason EF passes through G.

$\therefore$ EF bisects the two diagonals.

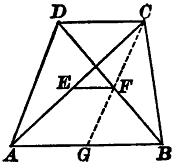
Ex. 62. The line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the bases.

Let $ABCD$ be a trapezoid, AB, DC the $\parallel$ sides, AC, BD its diagonals, E, F their middle points.

Then $EF = \frac{1}{2}(AB - DC)$.

PROOF. Draw CF, and produce it to meet AB at G.

In the ▲ BFG, CFD,



$BF = DF$. (Hyp.)

$\angle BFG = \angle CFD$, (§ 95)

$\angle GBF = \angle CDF$. (§ 104)

$\therefore \triangle BFG = \triangle CFD$, (§ 147)

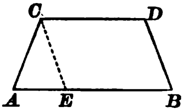
and $CF = FG$, $DC = BG$.

$\therefore AG = AB - BG = AB - DC$.

But $EF = \frac{1}{2} AG$ (§ 189). $\therefore EF = \frac{1}{2}(AB - DC)$.

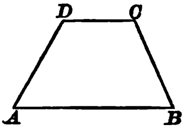
Ex. 63. In an isosceles trapezoid each base makes equal angles with the legs.

Let $ABCD$ be a trapezoid with leg $AC =$ leg BD.
Then $\angle A = \angle B$, and $\angle C = \angle D$.



Proof. Draw $CE \parallel$ to DB.
$CE = DB$. (§ 180)
$\therefore CE = AC$.
$\therefore \angle A = \angle CEA$. (§ 154)
But $\angle CEA = \angle B$ (§ 106). $\therefore \angle A = \angle B$.
Also $\angle C = 180^\circ - \angle A$ (§ 109), $\angle D = 180^\circ - \angle B$.
$\therefore \angle C = \angle D$.

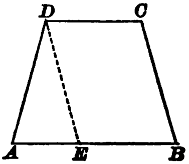
Ex. 64. In an isosceles trapezoid the opposite angles are supplementary.



Let $ABCD$ be an isosceles trapezoid,
and let AB be $\parallel$ to DC.
Then $\angle A + \angle C = 180^\circ = \angle B + \angle D$.
Proof. $\angle A + \angle D = 180^\circ$, (§ 109)
$\angle C = \angle D$. (Ex. 63)
$\therefore \angle A + \angle C = 180^\circ$.
Similarly, we may prove that $\angle B + \angle D = 180^\circ$.

Ex. 65. If the angles at the base of a trapezoid are equal, the other angles are equal, and the trapezoid is isosceles.

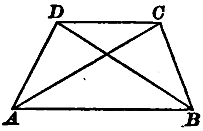
Let $ABCD$ be a trapezoid having $AB \parallel DC$ and $\angle A = \angle B$.
Then $\angle D = \angle C$, and $AD = BC$.



Proof. The $\angle$s D and C are equal, because they are the supplements of the equal $\angle$s A and B, respectively.
Draw $DE \parallel$ to CB.
Then $\angle DEA = \angle B$. (§ 106)
But $\angle B = \angle A$. (Hyp.)
$\therefore \angle DEA = \angle A$.
$\therefore AD = DE$. (§ 156)
But $DE = BC$ (§ 180). $\therefore AD = BC$.

Ex. 66. The diagonals of an isosceles trapezoid are equal.

Let $ABCD$ be an isosceles trapezoid, and AB, CD the ∥ sides, AC, BD the diagonals. Then $AC = BD$.

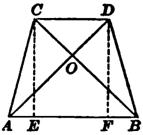


Proof. In the $\triangle$s ADC, BCD, DC is common.
$AD = BC$, (Hyp.)
$\angle ADC = \angle BCD$. (Ex. 63)
$\therefore \triangle ADC = \triangle BCD$, (§ 150)
and $AC = BD$.

Ex. 67. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

Let $ABCD$ be a trapezoid, AB, DC the ∥ sides, and let the diagonals AD, BC be equal.

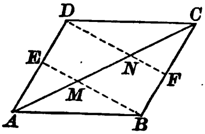
Then $AC = BD$, and the trapezoid is isosceles.



Proof. From C, D drop the $\perp$s CE, DF to AB.
CE is ∥ to DF. (§ 100)
$\therefore CE = DF$. (§ 180)
In the rt. $\triangle$s ADF, BCE,
$DF = CE$, and $AD = BC$. (Hyp.)
$\therefore \triangle ADF = \triangle BCE$ (§ 161). $\therefore \angle DAF = \angle CBA$.
In $\triangle$s ABC and ABD, AB is common.
$BC = AD$ (hyp.), and $\angle CBA = \angle DAF$.
$\therefore \triangle ABC = \triangle ABD$, and $AC = BD$.

Ex. 68. $ABCD$ is a parallelogram, E and F the midde points of AD and BC, respectively; show that BE and DF will trisect the diagonal AC.

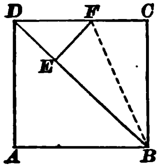
Let BE cut AC at M, and FD cut AC at N.



Then $AM = MN = NC$.

Proof. In the figure $EBFD$,
ED is ∥ to BF. (§ 168)
Also since $AD = BC$, (§ 179)
$\therefore ED = BF$. (Ax. 7)
$\therefore$ figure $EBFD$ is a ▱, (§ 182), and DF is ∥ to EB.
In the $\triangle ADN$, $AM = MN$. (§ 188)
And in the $\triangle CMB$, $CN = MN$. (§ 188)
$\therefore AM = MN = NC$.

Ex. 69. If from the diagonal BD of a square $ABCD$, BE is cut off equal to BC, and EF is drawn perpendicular to BD to meet DC at F, show that DE is equal to EF, and also to FC.



Proof. In the $\triangle BCD$,

$$\angle D = \angle B. \qquad (§\ 154)$$

But $\angle D + \angle B = 90°. \quad \therefore \angle D = 45°.$

In $\triangle DEF$, $\angle E + \angle D = 90° + 45° = 135°.$

$$\therefore \angle F = 180° - 135° = 45°.$$

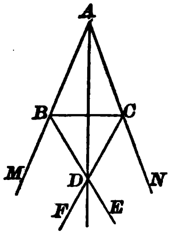
$\therefore \triangle DEF$ is isosceles, and $DE = EF$. (§ 156)

Join BF. In rt. $\triangle$s BEF, BCF, BF is common, and $BE = BC$. (Hyp.)

$$\therefore \triangle BEF = \triangle BCF \text{ (§ 161), and } EF = FC.$$

$$\therefore DE = EF = FC.$$

Ex. 70. The bisector of the vertical angle A of a triangle ABC, and the bisectors of the exterior angles at the base formed by producing the sides AB and AC, meet in a point which is equidistant from the base and the sides produced.



Let ABC be a $\triangle$, and produce AB to any point M, AC to any point N. Let also the bisectors BE, CF of the $\angle$s MBC, NCB meet at the point D.

Then bisector of $\angle A$ passes through D, and D is equidistant from AM, AN, and BC.

Proof. Every point in BE is equidistant from BM and BC (§ 162).

And every point in CF is equidistant from CN and BC (§ 162).

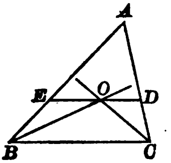
$\therefore$ the point D, which is common to BE and CF, is equidistant from BM, BC, and CN.

$\therefore$ the bisector of the $\angle A$ passes through D (§ 163).

Ex. 71. If the two angles at the base of a triangle are bisected, and through the point of meeting of the bisectors a line is drawn parallel to the base, the length of this parallel between the sides is equal to the sum of the segments of the sides between the parallel and the base.

Let ED be drawn $\parallel$ to the base BC of the $\triangle ABC$ through O, the intersection of the bisectors OB, OC of the $\angle$s ABC, ACB.

Then $$ED = EB + DC.$$



Proof. $\angle EBO = \angle OBC$, (Hyp.)

$\angle EOB = \angle OBC.$ (§ 104)

$\therefore \angle EBO = \angle EOB.$ (Ax. 1)

$\therefore \triangle OEB$ is isosceles,

and $EO = EB.$ (§ 156)

Similarly, we can prove that $OD = DC.$

Adding these two equations, $ED = EB + DC.$

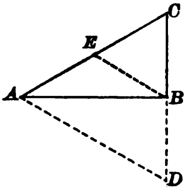
Ex. 72. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shortest side.

Let ABC be a rt. $\triangle$, right-angled at B, and let $\angle ACB = 2\angle CAB.$

Then $AC = 2BC.$

Proof. Produce CB to D, making BD equal to CB, and join AD.

In the rt. ⊿ ABD, ABC, AB is common, and $BD = BC.$ (Cons.)



$\therefore \triangle ABD = \triangle ABC.$ (§ 151)

$\therefore \angle ADC = \angle ACD,$

and $\angle DAB = \angle CAB.$

But $2\angle CAB = \angle ACD.$ (Hyp.)

$\therefore \angle DAC = \angle ACD.$

$\therefore \triangle ACD$ is equilateral. (§ 157)

$\therefore AC = CD = 2BC.$

Another Proof. Since $\angle A + \angle C = 90°$, (§ 143)

and $\angle C = 2\angle A.$

$\therefore \angle A = 30°$, and $\angle C = 60°.$

Suppose BE drawn, making the $\angle CBE = 60°.$

Then, since the ⦣ C and CBE are each 60°,

the $\angle CEB = 60°$, and $\triangle CBE$ is equilateral; (§ 157)

that is, the sides CB, CE, and BE are equal. (1)

Also, $\angle ABC = 90°$, and $\angle CBE = 60°.$

$\therefore \angle ABE = 90° - 60° = 30°.$

But $\angle A = 30°.$

Hence the sides AE and BE are equal (§ 156). (2)

From (1) and (2), $CB = CE = AE.$

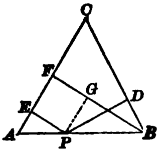
That is, $CB = \frac{1}{2}CA,$

or $AC = 2BC.$

Ex. 73. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the legs is constant, and equal to the altitude upon one of the legs.

Let ABC be an isosceles $\triangle$, AB the base, P any point in AB, $PD \perp$ to BC, $PE \perp$ to AC, and draw $BF \perp$ to the leg AC.

Then $\quad PD + PE = BF.$



Proof. Draw $PG \perp$ to BF; PE is $\parallel$ to BF, and PG is $\parallel$ to AC (§ 100).

$\therefore \angle BAC = \angle BPG$; (§ 106)

also $\quad \angle ABC = \angle BAC.$ (§ 154)

$\therefore \angle ABC = \angle BPG.$ (Ax. 1)

Also in the rt. $\triangle$ PBD, PBG, PB is common.

$\therefore \triangle PBD = \triangle PBG.$ (§ 148)

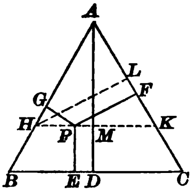
$\therefore PD = BG.$

Also $\quad PE = GF.$ (§ 180)

$\therefore PD + PE = BG + GF = BF.$

Ex. 74. The sum of the perpendiculars dropped from any point within an equilateral triangle to the three sides is constant, and equal to the altitude.

Let ABC be an equilateral $\triangle$, BC the base, AD the altitude, P any point within the $\triangle$, PE, PF, PG $\perp$s dropped from P to the sides BC, AC, AB, respectively. Then $PE + PF + PG = AD$.



Proof. Through P draw $HK \parallel$ to BC, meeting AD at M.

From H drop the $\perp$ HL to AC.

$\angle AHK = \angle ABC = 60°.$ (§ 106)

$\angle AKH = \angle ACB = 60°.$

$\therefore \triangle AHK$ is isosceles. (§ 156)

$\therefore PF + PG = HL.$ (Ex. 73)

In the rt. $\triangle$ AHL, AHM, the hypotenuse AH is common.

$\angle AHL = \angle HAM.$ (Ax. 7)

$\therefore \triangle AHL = \triangle AHM$, and $HL = AM$. (§ 148)

Also $\quad PE = MD.$ (§§ 100, 180)

$\therefore PE + HL = MD + AM = AD,$

or $\quad PE + PF + PG = AD.$

Ex. 75. What is the locus of all points equidistant from a pair of intersecting lines?

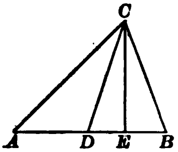
The locus consists of the two bisectors of the angles formed at the intersection of the given lines (§ 164).

These two bisectors are $\perp$ to each other (Ex. 7).

Ex. 76. In the triangle CAB the bisector of the angle C makes with the perpendicular from C to AB an angle equal to half the difference of the angles A and B. [$\angle B > \angle A$.

In the $\triangle ABC$, let CD bisect $\angle C$, and let CE be $\perp$ to AB, and let

Then $\angle DCE = \frac{1}{2}(\angle B - \angle A)$.



Proof. $\angle DCE = \angle ACE - \angle ACD$.

Now $\angle ACE = 90° - \angle A$. (§ 143)

And $\angle ACD = \frac{1}{2}\angle C$. (Hyp.)

$\therefore \angle DCE = 90° - \angle A - \frac{1}{2}\angle C$.

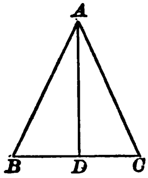
Also $\angle C = 180° - \angle A - \angle B$.

$\therefore \frac{1}{2}\angle C = 90° - \frac{1}{2}\angle A - \frac{1}{2}\angle B$.

$$\therefore \angle DCE = 90° - \angle A - (90° - \tfrac{1}{2}\angle A - \tfrac{1}{2}\angle B)$$
$$= 90 - \angle A - 90° + \tfrac{1}{2}\angle A + \tfrac{1}{2}\angle B$$
$$= \tfrac{1}{2}\angle B - \tfrac{1}{2}\angle A.$$

Ex. 77. If one angle of an isosceles triangle is equal to 60°, the triangle is equilateral.

Let ABC be an isosceles $\triangle$, and let $AB = AC$.



Case I. Let $\angle A = 60°$.

Then $\triangle ABC$ is equilateral.

Proof. $\angle A + \angle B + \angle C = 180°$. (§ 138)

$\therefore \angle B + \angle C = 180° - \angle A = 120°$.

But $\angle B = \angle C$. (§ 154)

$\therefore \angle B = 60°$, and $\angle C = 60°$.

$\therefore \triangle ABC$ is equilateral. (§ 155)

Case II. Let a base $\angle$, as $\angle B$, be 60°. Then $\triangle ABC$ is equilateral.

Proof. $\angle C = 60°$. (§ 154)

$\therefore \angle A = 180° - (\angle B + \angle C) = 60°$.

$\therefore \triangle ABC$ is equilateral. (§ 155)

Book II.

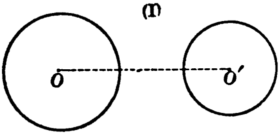
Ex. 78. Describe the relative position of two circles if the line of centres:

(i.) is greater than the sum of the radii;
(ii.) is equal to the sum of the radii;
(iii.) is less than the sum but greater than the difference of the radii;
(iv.) is equal to the difference of the radii;
(v.) is less than the difference of the radii.

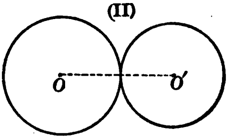
Illustrate each case by a figure.

These five cases are illustrated by the following figures:

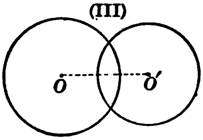
(I)



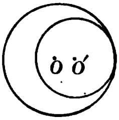
(II)



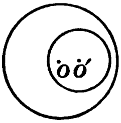
(III)



(IV)

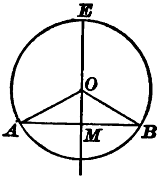


(V)



(i.) The two Ⓢ are exterior to one another.
(ii.) The two Ⓢ touch each other externally.
(iii.) The two Ⓢ intersect each other.
(iv.) The two Ⓢ touch each other internally.
(v.) The two Ⓢ lie one wholly within the other.

Ex. 79. The line joining the centre of a circle to the middle of a chord is perpendicular to the chord.



Let *OM* join the centre *O* to the middle point *M* of the chord *AB*.

Then *OM* is ⊥ to *AB*.

Proof. In the $\triangle s$ *AOM*, *BOM*, *OM* is common,

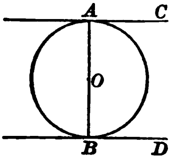
$AM = BM$ (hyp.), and $OA = OB$ (radii).

$\therefore \triangle AOM = \triangle BOM$, (§ 160)

and $\angle AMO = \angle BMO$.

$\therefore OM$ is ⊥ to AB. (§§ 45, 47)

Ex. 80. The tangents drawn through the extremities of a diameter are parallel.



Let AB be a diameter of a circle, AC and BD tangents at A and B, respectively.

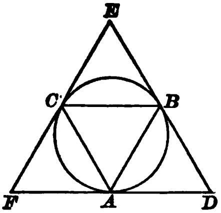
Then AC is ‖ to BD.

Proof. AC and BD are each ⊥ to AB. (§ 240)

∴ AC is ‖ to BD. (§ 100)

Ex. 81. The perimeter of an inscribed equilateral triangle is equal to half the perimeter of the circumscribed equilateral triangle.

Let △ EDF be a circumscribed equilateral △, touching the circumference at B, A, C.



Draw AB, AC, and CB.

Then perimeter of △ ABC equals $\frac{1}{2}$ perimeter of △ EDF.

Proof. Now each ∠ of △ $EDF = 60°$, (§ 155) and $EB = EC$, $DB = DA$, $FA = FC$. (§ 246)

∴ ▲ EBC, DBA, FAC are equilateral. (Ex. 77)

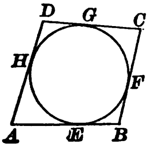
Also, since each of the ∠s of the △ ABC is the difference between a straight ∠ and the sum of two ∠s of equilateral ▲, each ∠ of △ ABC is 60°.

∴ △ ABC is equiangular and equilateral (§ 157); and since △ ABC has a side in common with each of the ▲ EBC, DBA, and FAC, those four ▲ are equal (§ 147), and a side of △ $ABC = \frac{1}{2}$ a side of △ EDF.

Hence the perimeter of △ $ABC = \frac{1}{2}$ the perimeter of △ EDF.

Ex. 82. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Let $ABCD$ be a circumscribed quadrilateral, and let E, F, G, H, be the points of contact.



Then $AB + DC = AD + BC$.

Proof. $AE = AH$, (§ 246)

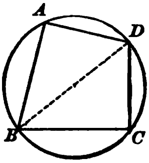
$BE = BF$,

$CG = CF$,

$DG = DH$.

Adding, $AB + DC = AD + BC$.

Ex. 83. The opposite angles of an inscribed quadrilateral are supplements of each other.



Let $ABCD$ be an inscribed quadrilateral.

Then $\angle A + \angle C = \angle B + \angle D = 180°$.

Proof. Join BD.

Then $\angle A$ is measured by $\frac{1}{2}$ arc DCB,

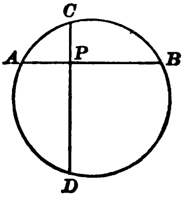
and $\angle C$ is measured by $\frac{1}{2}$ arc DAB.

$\therefore$ the sum, $\angle A + \angle C$, is measured by $\frac{1}{2}$(arc DCB + arc DAB), or $\frac{1}{2}$ the circumference.

$\therefore \angle A + \angle C = \frac{1}{2}(360°) = 180°$.

Similarly, we can prove that $\angle B + \angle D = 180°$.

Ex. 84. If through a point within a circle two perpendicular chords are drawn, the sum of the opposite arcs which they intercept is equal to a semi-circumference.



Let AB and CD be two $\perp$ chords meeting at P.

Then arc AC + arc BD = a semi-circumference.

Proof. $\frac{1}{2}$(arc AC + arc BD) measures $\angle APC$ (§ 268).

But $\angle APC = 90°$.

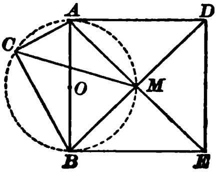
$\therefore \frac{1}{2}$(arc AC + arc BD) $= 90°$.

$\therefore$ arc AC + arc BD $= 180°$ (measured in degrees)

= a semi-circumference.

Ex. 85. The line joining the centre of the square described upon the hypotenuse of a right triangle to the vertex of the right angle, bisects the right angle.

Let ABC be a rt. Δ, and $\angle C = 90°$; also let M be the centre of the square $ABED$ constructed upon the hypotenuse AB; in other words, let M be the intersection of the two diagonals.



Then $\angle ACM = \angle BCM$.

Proof. Describe a ⊙ upon AB as diameter.

$\angle ACB = 90°$, (Hyp.)

and $\angle AMB = 90°$. (Ex. 29)

$\therefore$ the ⊙ will pass through C and M. (§ 264)

[illegible]

Now $\angle MAB = \angle MBA$. (Ex. 29.) $\therefore AM = BM$. (§ 156)

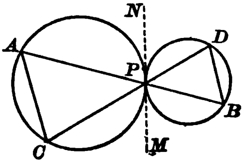
$\therefore$ arc $AM =$ arc BM. (§ 230.) $\therefore \angle ACM = \angle BCM$. (§ 263)

And since $\angle ACB = 90°$, $\angle ACM = \angle BCM = 45°$.

Ex. 86. If two circles touch each other and two secants are drawn through the point of contact, the chords joining their extremities are parallel.

Let the ⑤ touch at P, and let AB and CD be the two secants.

Then AC is ∥ to DB.



Proof. Draw through P the common tangent MN.

Then $\angle A = \angle MPC$,

and $\angle B = \angle NPD$. (§§ 263, 269)

But $\angle MPC = \angle NPD$. (§ 95)

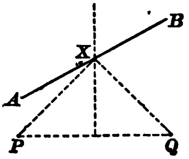
$\therefore \angle A = \angle B$. $\therefore AC$ is ∥ to DB. (§ 105)

Ex. 87. To find in a given line a point X which shall be equidistant from two given points.

Let AB be the given line, P, Q the given points. Join PQ, and erect the perpendicular bisector, cutting AB at X. (§ 273)

Then X will be the point required.

Proof. $PX = QX$. (§ 122)



Ex. 88. To find a point X which shall be equidistant from two given points and at a given distance from a third given point.

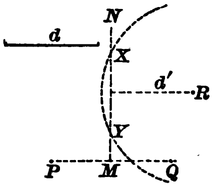
Let P, Q be the given points, d the given distance, R the third point.

Join PQ, and erect the perpendicular bisector MN.

With centre R and radius d, describe an arc cutting MN at X.

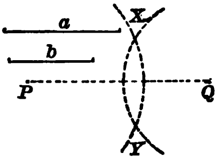
Then X will be the point required.

Proof. X is equidistant from P and Q, by § 122, and at the distance d from R, by construction.



Discussion. Let d' denote the distance from R to MN. Then the number of solutions will be 2, 1, or 0, according as d is $>$, $=$, or $< d'$. In the figure there are two solutions, X and Y.

Ex. 89. To find a point X which shall be at given distances from two given points.



Let P, Q be the given points, and let a, b be the given distances.

With centre P and radius a, describe an arc.

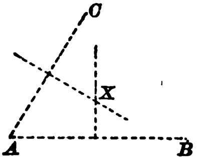
With centre Q and radius b, describe another arc cutting the first arc at X.

Then X is the point required.

Proof. X is, by construction, at the distance a from P, and at the distance b from Q.

Discussion. If $PQ = a + b$, there is one solution; namely, a point in the line PQ where the two arcs touch each other. If PQ is $> a + b$, or $< a - b$, the problem is impossible; for the arcs cannot cut each other and cannot touch each other. (See Ex. 78.) If PQ has any value between $a + b$ and $a - b$, there are two solutions (as in the figure).

Ex. 90. To find a point X which shall be equidistant from three given points.



Let A, B, C be the given points.

Join AB, AC, and erect upon each line the perpendicular bisector. Let these perpendicular bisectors meet at X.

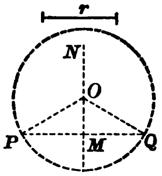
Then X is equidistant from A, B, and C.

Proof. Join XA, XB, XC.

Then $$XA = XB = XC. \quad (\S\ 122)$$

Discussion. The problem is impossible if the points A, B, C are in a straight line.

Ex. 91. To construct a circle having a given radius and passing through two given points.



Let P, Q be the given points, r the given radius. Join PQ, and draw the perpendicular bisector MN.

With centre P and radius r, cut MN at O.

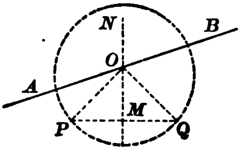
With centre O and radius r, describe a ⊙.

This ⊙ will be the ⊙ required.

Proof. The ⊙ has r for radius, and passes through P, Q, by § 122.

Discussion. The arc described with centre P and radius r will also cut MN in a point below PQ; hence two ⓢ answering the conditions may be described. These ⓢ, however, are equal, and differ merely in position. This problem is impossible if r is $< \frac{1}{2} PQ$.

Ex. 92. To construct a circle having its centre in a given line and passing through two given points.



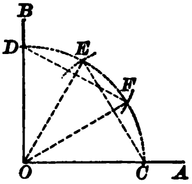
Let AB be the given line, P and Q the given points.

Join PQ, and draw the perpendicular bisector MN. Let MN meet AB at O.

With centre O and radius OP, describe a ⊙. This ⊙ will be the ⊙ required.

Proof. The ⊙ passes through P and Q, by § 122.

Ex. 93. To divide a right angle into three equal parts.



Let $\angle AOB$ be a right angle.

With centre O and any radius, describe an arc cutting OA at C, and OB at D.

With centres C and D, and the same radius as before, cut arc DC at E and F respectively.

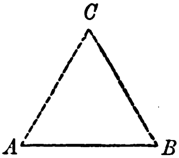
Then will $\angle COF = \angle FOE = \angle EOD = 30°$.

Proof. ⚠ OCE, OFD, are equilateral by construction.

$\therefore \angle COE = 60°$ (§ 155). $\therefore \angle EOD = 90° - \angle COE = 30°$.

Similarly, $\angle COF = 30°$. $\therefore \angle FOE = 90° - 60° = 30°$.

Ex. 94. To construct an equilateral triangle, having given one side.



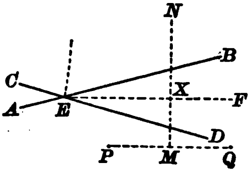
Let AB be the given side.

With centres A and B and radius AB, describe arcs intersecting in C.

Join AC, BC.

Then $\triangle ABC$ is the triangle required, as is obvious from the construction.

Ex. 95. To find a point X which shall be equidistant from two given points, and also equidistant from two given intersecting lines.



Let P, Q be the given points, and let AB, CD be the given lines intersecting at E.

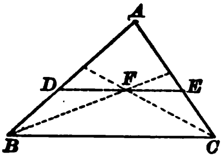
Join PQ, and erect the perpendicular bisector MN (§ 273).

Draw EF bisecting $\angle DEB$ (§ 275), and meeting MN at X. Then X is the point required.

Proof. X is equidistant from P and Q (§ 122), and also equidistant from AB and CD (§ 162).

DISCUSSION. By bisecting the $\angle BEC$, another solution Y may be obtained. Therefore, in general there are two solutions. If, however, EF happens to coincide with PQ, the other bisector EY will be ∥ to MN (§ 100), and there will be only one solution; namely, the point M. If EF coincides with MN, every point in MN is a solution.

Ex. 96. In a triangle ABC, to draw DE parallel to the base BC, cutting the sides of the triangle in D and E, so that DE shall equal $DB + EC$.



Bisect ∠s B and C (§ 275), and let the bisectors meet at F.

Draw through F a line ∥ to BC, meeting AB at D, and AC at E.

Then $DE = BD + EC$.

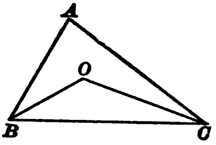
PROOF. In $\triangle BDF$,

$\angle DFB = \angle FBC$. (§ 104) Also $\angle DBF = \angle FBC$. (Cons.)

$\therefore \angle DBF = \angle DFB$. $\therefore DF = DB$. (§ 156)

Similarly, $FE = EC$. $\therefore DF + FE$ or $DE = DB + EC$.

Ex. 97. If an interior point O of a triangle ABC is joined to the vertices B and C, the angle BOC is greater than the angle BAC of the triangle.



Let O be any point within the $\triangle ABC$.

Then $\angle BOC > \angle BAC$.

PROOF. $\angle OBC < \angle ABC$.

$\angle OCB < \angle ACB$.

$\therefore \angle OBC + \angle OCB < \angle ABC + \angle ACB$.

Now $\angle O = 180° - \angle OBC - \angle OCB$. (§ 139)

And $\angle A = 180° - \angle ABC - \angle ACB$.

$\therefore \angle O > \angle A$.

Ex. 98. In a triangle ABC, given angles A and B, equal respectively to 37° 13′ 32″ and 41° 17′ 56″. Find the value of $\angle C$.

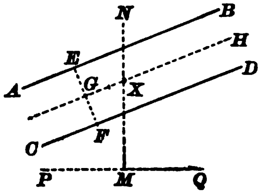
Ans. $\angle C = 180° - (37° 13' 32'' + 41° 17' 56'') = 101° 28' 32''$.

Ex. 99. To find a point X equidistant from two given points and also equidistant from two given parallel lines.

Let P, Q be the points, AB and CD the parallels. Join PQ, and erect the perpendicular bisector MN.

Draw $EF \perp$ to AB (§ 271), bisect EF at G (§ 273), and draw $GH \parallel$ to AB (§ 278), meeting MN at X.

Then X is the point required.



PROOF. Every point in MN is equidistant from P and Q (§ 122), and every point in GH is equidistant from AB and CD. (§ 181)

$\therefore X$, the point common to MN and GH, is equidistant from P, Q, AB, and CD.

DISCUSSION. In general, there is one solution. If AB is $\perp$ to PQ, there is no solution, unless GH coincides with MN; and then every point in MN is a solution.

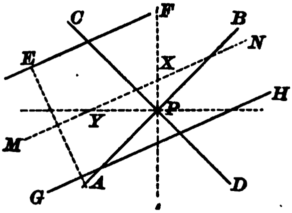
Ex. 100. To find a point X equidistant from two given intersecting lines, and also equidistant from two given parallels.

Let AB, CD be the intersecting lines, P the point of intersection, EF and GH the two parallel lines.

Draw $MN \parallel$ to EF and GH, and midway between them (§§ 271, 273, 278).

Also draw the bisectors of the angles formed at P (§ 275). Let these bisectors meet MN at X and Y.

Then X and Y satisfy the conditions.



PROOF. X and Y are equidistant from EF and GH because they are in MN, and equidistant from AB and CD because they are in the bisectors of the angles formed by AB and CD (§ 162).

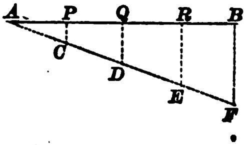
DISCUSSION. If MN passes through P, there is only one solution; namely, the point P. If MN is $\parallel$ to one of the bisectors, there is but one solution. If MN coincides with one of the bisectors, every point in MN is a solution.

Ex. 101. To divide a line into four equal parts by two different methods.

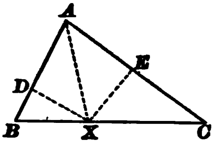
Let AB be the given line.

First Method. Draw AF, and proceed as in § 279.

Second Method. Bisect the line at Q (§ 273), and then bisect AQ at P, BQ at R.



Ex. 102. To find a point X in one side of a given triangle and equidistant from the other two sides.



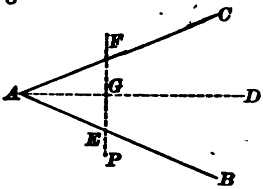
Let ABC be the given triangle.

Bisect the $\angle A$ (§ 275), and let the bisector meet BC at X.

Then X is equidistant from AB and AC.

Proof. See § 162.

Ex. 103. Through a given point to draw a line which shall make equal angles with the two sides of a given angle.



Let P be the point, BAC the angle.

Draw AD bisecting the $\angle BAC$. (§ 275)

Through P draw $EF \perp$ to AD, and meeting AB at E, AC at F. (§ 272)

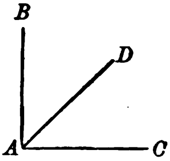
EF is the line required.

Proof. Let EF meet AD at G.

In $\triangle$ AEG, AFG, AG is common, and $\angle EAG = \angle FAG$. (Cons.)

$\therefore \triangle AEG = \triangle AFG$ (§ 149). $\therefore \angle AEG = \angle AFG$.

Ex. 104. To construct an angle of 45°.



Construct a rt. $\angle BAC$. (§ 271)

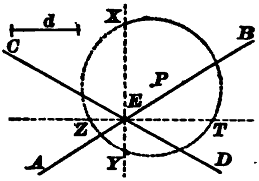
Draw AD bisecting the $\angle BAC$. (§ 275)

Then $\angle BAD = \angle DAC = 45°$.

Proof. A rt. $\angle = 90°$, and $90° + 2 = 45°$.

Ex. 105. To find a point X which shall be equidistant from two given intersecting lines and at a given distance from a given point.

Let AB, CD be the given lines, intersecting at E, d the given distance, P the given point.



Draw the bisectors XY, TZ of the angles formed by the two lines AB, CD. (§ 275)

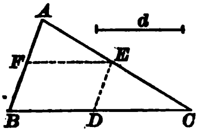
With centre P and radius d, describe an arc cutting the bisectors in X, Y, Z, T.

Then each of the points X, Y, Z, T satisfies the conditions.

Proof. Each of the points is equidistant from the given lines, by § 162, and at the distance d from P by construction.

Discussion. If (as in the figure) d is greater than the distance from P to each bisector, there are four solutions. If d is less than the distance from P to each bisector, there is no solution. Between these extremes there may be one, two, or three solutions.

Ex. 106. To draw through two sides of a triangle a line parallel to the third side so that the part intercepted between the sides shall have a given length.



Let ABC be the given $\triangle$, d the given length. Upon BC take BD equal to d.

Draw $DE \parallel$ to BA, meeting AC at E. (§ 278)

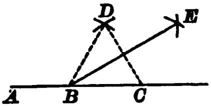
Draw $EF \parallel$ to CB, meeting AB at F.

Then EF is the line required.

Proof. EF is $\parallel$ to CB. (Cons.) The figure $BDEF$ is a ▱. (§ 168) $EF = BD$. (§ 179). But $BD = d$. (Cons.) $\therefore EF = d$.

Ex. 107. To construct an angle of 150°.

Draw a straight line ABC, and with two points in it B, C, as centres, and BC for radius, describe arcs intersecting at D.



With centres D and C, and the same radius, describe arcs intersecting at E.

Join BE. $\angle ABE = 150°$.

Proof. Join BD, CD. $\triangle BCD$ is equilateral. (Cons.)

$\therefore \angle DBC = 60°$. (§ 155)

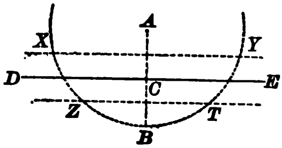
$\therefore \angle ABD = 180° - 60° = 120°$. (§ 90)

$\angle DBE = \frac{1}{2} \angle DBC = 30°$. (§ 275)

$\therefore \angle ABE = 120° + 30° = 150°$.

Ex. 108. A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. To find by construction how many places answer this description.

Let A represent the town. With centre A and any radius describe a circle.



Draw a radius AB; and take this length to represent 4 miles.

Construct the perpendicular bisector DE to the line AB, bisecting it at C.

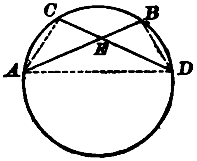
Then DE will represent the railroad.

Draw parallels to DE on either side, at a distance from DE equal to $\frac{1}{2} AC$ (1 mile).

Let these parallels cut the circle in the points X, Y, Z, T. Then any one of these four points satisfies the conditions.

Therefore there are four solutions.

Ex. 109. If in a circle two equal chords intersect, the segments of the one made by the point of intersection are equal respectively to those of the other.



Let AB, CD be the equal chords intersecting at the point E.

Then $AE = DE$, and $BE = CE$.

PROOF. Join AC, AD, BD.

In the $\triangle$ ABD, ACD, AD is common.

Also, $AB = CD$. (Hyp.)

$\therefore$ arc ACB = arc CBD. (§ 230)

Subtract, arc BC = arc BC.

Then arc AC = arc BD.

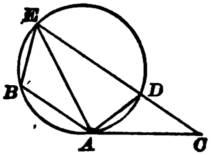
$\therefore AC = BD$. (§ 230)

$\therefore \triangle ABD = \triangle ACD$ (§ 160), and $\angle BAD = \angle CDA$.

$\therefore$ in the $\triangle AED$, $AE = DE$. (§ 156)

Also $AB - AE = CD - DE$, or $BE = CE$.

Ex. 110. AB is any chord and AC is tangent to a circle at A, CDE a line cutting the circumference in D and E and parallel to AB; show that the triangles ACD and EAB are mutually equiangular.



PROOF. $\angle BAE$ is measured by $\frac{1}{2}$ arc BE (§ 263).

$\angle CAD$ is measured by $\frac{1}{2}$ arc AD. (§ 269)

And arc BE = arc AD. (§ 243)

$\therefore \angle BAE = \angle CAD$.

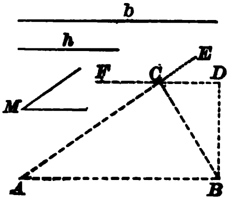
Also $\angle AEB$ is measured by $\frac{1}{2}$ arc AB. (§ 263)

And $\angle ACD$ is measured by $\frac{1}{2}$ (arc AE − arc AD) (§ 270)

Or (since arc BE = arc AD), by $\frac{1}{2}$ (arc AE − arc BE), or $\frac{1}{2}$ arc AB.

$\therefore \angle AEB = \angle ACD$. $\therefore \angle ABE = \angle ADC$. (§ 140)

Ex. 111. The base, the altitude, and an angle at the base, of a triangle being given, to construct the triangle.



Let b be the given base, h the given altitude, and M the given angle at the base.

Draw a line AB equal to b.

Make an $\angle BAE$ equal to M. (§ 276)

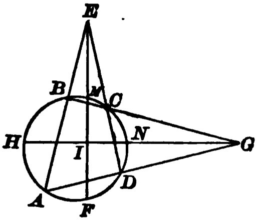
Erect at B a $\perp$ BD equal to h. (§ 271)

Draw through D a line $DF \parallel$ to BA. (§ 278)

Let this parallel cut AE at C. Join BC. $\triangle ABC$ is the triangle required.

Proof. $\triangle ABC$ has the parts b and M by direct construction, and its altitude equals h by construction and § 181.

Ex. 112. Show that the bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.



Let $ABCD$ be the inscribed quadrilateral, and let AB and DC meet at E, and AD and BC meet at G. Also let the bisectors EF, GH of the $\angle$s at E and G meet at I. Then EF is $\perp$ to GH.

Proof. Let the bisector of $\angle E$ cut the $\odot$ at M and F, and the bisector of the $\angle G$ cut the $\odot$ at N and H. Then, since $\angle AEF = \angle DEF$, (Hyp.)

$$\tfrac{1}{2}(\text{arc } AF - \text{arc } BM) = \tfrac{1}{2}(\text{arc } DF - \text{arc } CM). \qquad (§ 270)$$

$$\therefore \text{arc } AF - \text{arc } BM = \text{arc } DF - \text{arc } CM.$$

Similarly, $\text{arc } AH - \text{arc } DN = \text{arc } BH - \text{arc } CN.$

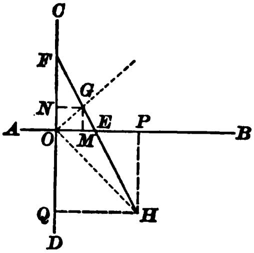
Adding, $\text{arc } FH - \text{arc } BM - \text{arc } DN = \text{arc } DF + \text{arc } BH - \text{arc } MN.$

Or $\text{arc } FH + \text{arc } MN = \text{arc } HM + \text{arc } FN.$

Hence the measure of the $\angle$s FIH, HIM is the same. (§ 268)

$$\therefore \angle FIH = \angle HIM = 90°, \text{ and } EF \text{ is } \perp \text{ to } GH.$$

Ex. 113. Given two perpendiculars, AB and CD, intersecting in O, and a straight line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)



Bisect the $\angle$s at O, and let one bisector meet EF at G, and the other meet EF produced at H. Through G draw $GM \parallel$ to CD, and $GN \parallel$ to BA. Through H draw $HP \parallel$ to DC, and $HQ \parallel$ to BA. The figures $OMGN$, $OPHQ$ are squares satisfying the given conditions.

Proof. In the figure $OMGN$, $\angle O = 90°$ (hyp.), $\angle$s at M and N each $= 90°$ (§ 102); $\angle MGN = \angle MON$ (§ 112). $\therefore \angle MGN = 90°$.

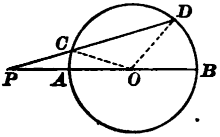
Also $GN = GM$ (§ 162), and $GN = OM$, and $GM = ON$ (§ 180).

$\therefore$ the figure $OMGN$ is a square (§ 171).

Similarly, the figure $OPHQ$ is proved to be a square.

Ex. 114. The shortest line and the longest line which can be drawn from an exterior point to a given circumference pass through the centre.

Let O be the centre of the given ⊙, P the given point, PAB a secant passing through O, PCD any other secant.



Then $PA < PC$ and $PB > PD$.

Proof. Join OC, OD.

Since $PA + AO < PC + CO$, (§ 33)

and $AO = CO$,

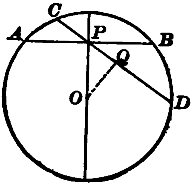
$\therefore PA < PC$. (Ax. 5)

Also, since $PO + OD > PD$, (§ 33)

and $OD = OB$,

$\therefore PO + OB$ or $PB > PD$.

Ex. 115. If through a point within a circle a diameter and a chord ⊥ to the diameter are drawn, the chord is the shortest chord that can be drawn through the given point.



Let P be the given point, O the centre, AB a chord passing through P and ⊥ to OP, CD any other chord drawn through P.

Then $AB < CD$.

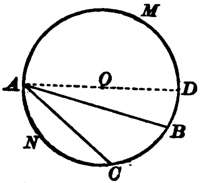
Proof. Draw OQ ⊥ to CD, meeting CD at Q.

Then in the rt. △ OPQ, $OP > OQ$. (§ 159)

$\therefore AB < CD$. (§ 238)

Ex. 116. In the same circle, or in equal circles, if two arcs are each greater than a semi-circumference, the greater arc subtends the *less* chord, and conversely.

Let O be the centre, AMB the smaller arc, $AMBC$ the larger arc, and let each arc be greater than the semi-circumference AMD. Then $AC < AB$.



Proof. AC is also the chord of the arc ANC, and AB the chord of the arc ANB.

And arcs ANC, ANB, are each less than a semi-circumference.

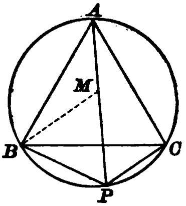
Also, since arc $ANC =$ circumference $-$ arc $AMBC$,

and arc $ANB =$ circumference $-$ arc AMB;

$\therefore$ arc $ANC <$ arc ANB.

$\therefore AC < AB$. (§ 231)

Ex. 117. If ABC is an inscribed equilateral triangle, and P is any point in the arc BC, then $PA = PB + PC$.



Proof. Upon PA take PM equal to PB.

Join BM.

$\angle BPM = 60°$. (§ 263)

$\therefore \triangle BPM$ is equilateral. (Ex. 77)

In the $\triangle$ ABM, BCP,

$AB = BC$, and $BM = BP$.

Also $\angle ABM = \angle ABP - \angle MBP = \angle ABP - 60°$.

$\angle PBC = \angle ABP - \angle ABC = \angle ABP - 60°$.

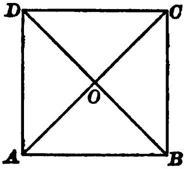
$\therefore \angle ABM = \angle PBC$.

$\therefore \triangle ABM = \triangle BCP$ (§ 150), and $AM = PC$.

$\therefore PA = PM + MA = PB + PC$.

Ex. 118. In what kinds of parallelograms can a circle be inscribed? Prove your answer correct.

Ans. The square and the rhombus.



Proof. In the square $ABCD$ draw the diagonals AC, BD, and let them meet at O.

AC and BD bisect the $\angle$ of the square. (Ex. 29)

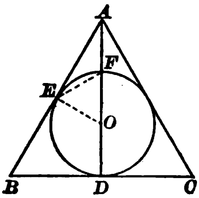
$\therefore$ the point O is equidistant from the sides of the square. (§ 162)

$\therefore$ a ⊙ with centre O and radius equal to the distance from O to one side will touch all the sides, and be inscribed in the square.

Similarly, it may be proved that a ⊙ can be inscribed in a rhombus. (See Ex. 28.)

Ex. 119. The radius of the circle inscribed in an equilateral triangle is equal to one-third of the altitude of the triangle.

Let ABC be an equilateral $\triangle$, O the centre of the inscribed ⊙, AOD the altitude of the $\triangle$. Then $OD = \frac{1}{3} AD$.



Proof. Let AB touch the ⊙ at E, and AD cut the ⊙ at F. Join OE, EF.

$\angle OEA = 90°$ (§ 240), and $\angle OAE = 30°$ (§ 246).

$\therefore \angle EOF = 60°$ (§ 143). And $OE = OF$.

$\therefore \triangle OEF$ is equilateral. (Ex. 77)

In $\triangle AEF$, $\angle AEF = 30°$. (§ 269)

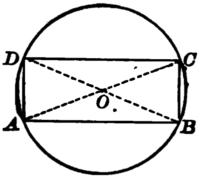
$\therefore AF = EF$ (§ 156). $\therefore OE = EF = AF$.

Let x denote the radius of the ⊙; then

$$x = \tfrac{1}{2} FD = \tfrac{1}{2}(AD - x), \text{ whence } x = \tfrac{1}{3} AD.$$

Ex. 120. A circle can be circumscribed about a rectangle.

Let $ABCD$ be a rectangle. Then a ⊙ can be described which will pass through the points A, B, C, and D.



Proof. Let the diagonals AC, BD intersect at O. AC and BD are equal. (Ex. 26)

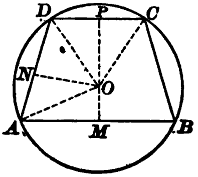
They also bisect each other. (§ 184)

∴ O is equidistant from A, B, C, and D.

∴ a ⊙ with centre O and radius OA will pass through the vertices and be circumscribed about the rectangle.

Ex. 121. A circle can be circumscribed about an isosceles trapezoid.

Let $ABCD$ be an isosceles trapezoid, and $AD = BC$. Then a ⊙ can be described which will pass through the points A, B, C, and D.



Proof. Erect the ⊥ bisectors of AB and AD, and let them meet at the point O.

Then O is equidistant from A, B, and D (§ 122).

Now $AM = BM$, $AD = BC$, and $\angle A = \angle B$ (Ex. 63).

∴ if the figure $AMPD$ be folded over PM as an axis, it will fall on and coincide with the figure $MBCP$.

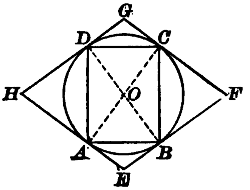
∴ $PD = PC$. Also PM is ⊥ to DC. (§ 102)

∴ $OC = OD$, and O is equidistant from A, B, C, and D.

∴ a ⊙ with centre O and radius OA will pass through the vertices, and be circumscribed about the trapezoid.

Ex. 122. The tangents drawn through the vertices of an inscribed rectangle enclose a rhombus.

Let O be the centre, $ABCD$ the inscribed rectangle, and let the tangents drawn through A, B, C, and D meet in the points E, F, G, and H.



Then the figure $EFGH$ is a rhombus.

Proof. The four ▲ in the figure are isosceles. (§ 246)

Since $AD = BC$, (§ 179)

arc AD = arc BC. (§ 230)

And $\angle A = \angle D = \angle B = \angle C$. (§ 269)

∴ $\triangle AHD = \triangle BFC$. (§ 147)

Similarly, $\triangle AEB = \triangle DGC$.

Hence by addition, $EF = FG = GH = HE$.

Now the figure $EFGH$ cannot be a square.

For, suppose it to be a square, and draw AC, BD, intersecting at the centre O. (Ex. 120)

$\angle G = 90°$, and also $\angle ODG = \angle OCG = 90°$. (§ 240)

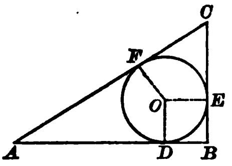
$\therefore \angle DOC = 90°$. (§ 206). $\therefore \angle AOD = 90°$. (§ 90)

$\therefore \triangle OAD = \triangle OCD$ (§ 150), and $AD = DC$.

This contradicts the hypothesis that $ABCD$ is a rectangle.

$\therefore$ the figure $EFGH$ must be a rhombus.

Ex. 123. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.



Let ABC be a rt. $\triangle$, $\angle B = 90°$, O the centre of the inscribed $\odot$, D, E, F the points of contact. Then diameter of $\odot = AB + BC - AC$.

Proof. Draw the radii OD, OE.

OD is $\perp$ to AB, and OE is $\perp$ to BC. (§ 240)

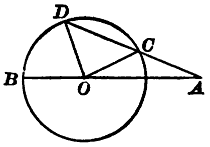
$\therefore OD$ is $\parallel$ to CB, and OE is $\parallel$ to AB. (§ 100)

$\therefore OD = BE$, and $OE = BD$. (§ 180)

$$\begin{aligned}\therefore OD + OE &= BE + BD\\ &= (BC - EC) + (AB - AD)\\ &= BC - FC + AB - AF \quad (§\ 246)\\ &= AB + BC - (AF + FC)\\ &= AB + BC - AC.\end{aligned}$$

$\therefore$ diameter of $\odot = AB + BC - AC$.

Ex. 124. From a point A without a circle, a line AOB is drawn through the centre, and also a secant ACD, so that the part AC without the circle is equal to the radius. Prove that the $\angle DAB$ equals one-third of the $\angle DOB$.



Proof. $AC = OC$. (Hyp.)

$\therefore \angle CAO = \angle COA$. (§ 154)

And $OC = OD$. $\therefore \angle ODC = \angle OCD$.

But $\angle OCD = 2\angle CAO$. (§ 145)

$\therefore \angle ODC = 2\angle CAO$.

Now $\angle DOB = \angle CAO + \angle ODC = 3\angle CAO$. (§ 145)

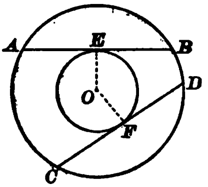
$\therefore \angle DAB = \frac{1}{3}\angle DOB$.

CE = CF and AF = AD

∴ rem. dif. = BD + BE = OD + OE = d

Ex. 125. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the points of contact.

Let O be the common centre of the Ⓢ, AB, CD any two chords of the outer ⊙, touching the inner circle at the points E, F. Then $AB = CD$, and $AE = BE$, $CF = DF$.



Proof. Join OE, OF.

OE is $\perp$ to AB, and OF is $\perp$ to CD. (§ 240)

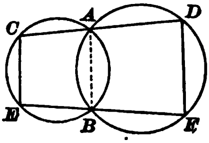
$AB = CD$. (§ 236)

$AE = BE$, and $CF = DF$. (§ 232)

Ex. 126. If two circles intersect and a secant is drawn through each point of intersection, the chords which join the extremities of the secants are parallel.

Let the Ⓢ intersect in the points A, B, and let CAD, EBF be any two chords drawn through A, B, respectively.

Then CE is $\parallel$ to DF.



Proof. Join AB. The figures $ACEB$, $ADFB$ are inscribed quadrilaterals.

$\therefore \angle E = 180° - \angle BAC$, (Ex. 83)

$\angle F = 180° - \angle BAD$.

But $\angle BAC = 180° - \angle BAD$. (§ 90)

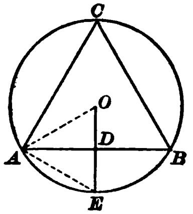
$\therefore \angle BAC = \angle F$, and $\angle E = 180° - \angle F$.

$\therefore \angle E + \angle F = 180°$, and CE is $\parallel$ to DF. (§ 110)

Ex. 127. If an equilateral triangle is inscribed in a circle, the distance of each side from the centre of the circle is equal to half the radius of the circle.

Let ABC be an inscribed equilateral $\triangle$, O the centre of the ⊙, OE a radius $\perp$ to AB, and meeting the side AB at D.

Then $OD = \frac{1}{2} OE$.



Proof. Join OA and AE.

arc AB = arc BC = arc AC. (§ 230)

$\therefore$ arc AB is an arc of 120°.

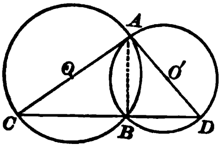
$\therefore$ arc $AE = \frac{1}{2}$ arc $AB = 60°$. (§ 232)

And since $OA = OE$, $\triangle OAE$ is equilateral. (Ex. 77)

$\therefore AE = OE$.

$\therefore OD = \frac{1}{2} OE$. (Ex. 16)

Ex. 128. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the straight line joining the ends of the diameters passes through the other point of intersection.



Let O, O' be the centres of the ⊙s; A and B the points of intersection, AC, AD two diameters.

Then a straight line from C to D will pass through B.

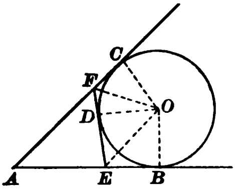
Proof. Join AB, BC, BD.

The △s ABC, ABD are rt. △s. (§ 264)

∴ CBD is a straight line. (§ 93)

∴ a straight line joining C to D will pass through B. (§ 31)

Ex. 129. A circle touches two sides of an angle BAC at B, C; through any point D in the arc BC a tangent is drawn, meeting AB at E and AC at F. Prove (i.) that the perimeter of the triangle AEF is constant for all positions of D; (ii.) that the angle EOF is also constant.



Proof. (i.) $ED = EB$, and $FD = FC$. (§ 246)

Adding, $EF = EB + FC$.

Also, $AE + AF = AE + AF$.

Adding, $AE + AF + EF = AB + AC$.

And $AB + AC$ is constant for all positions of the point D along the arc BC.

(ii.) $\angle EOD = \frac{1}{2} \angle BOD$, and $\angle FOD = \frac{1}{2} \angle COD$. (§ 246)

$$\therefore \angle EOD + \angle FOD = \tfrac{1}{2}(\angle BOD + \angle COD),$$

or $$\angle EOF = \tfrac{1}{2} \angle BOC.$$

And $\angle BOC$ is constant for all positions of D.

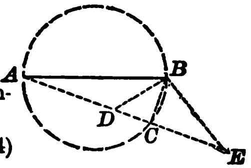
Ex. 130. Find the locus of a point 3 inches from a given point.

Ans. The circumference of a ⊙ described with the given point for centre, and with a radius equal to 3 inches.

Ex. 131. Find the locus of a point at a given distance from a given circumference.

Ans. Let r denote the radius of the given circumference, O its centre, d the given distance. Then the locus consists of the two circumferences, having for centre the point O, and for radii $r + d$ and $r - d$, respectively. If $d = r$, the second of these circumferences becomes the point O; if $d > r$, the second circumference does not exist.

Ex. 132. Prove that the locus of the vertex of a right triangle, having a given hypotenuse as base, is the circumference described upon the given hypotenuse as diameter.



Let AB be the given hypotenuse.

With AB as diameter, describe a ⊙.

PROOF. (i.) Join any point C of the circumference to A and B.

Then $\angle ACB = 90^\circ$. (§ 264)

(ii.) Join any point D in AC to B.

Then $\angle ADB > \angle ACB$, or 90°. (§ 146)

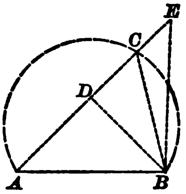
Produce AC to any point E, and join EB.

Then $\angle AEB < \angle ACB$, or 90°. (§ 146)

Therefore the circumference with AB for diameter is the locus of the vertex of a rt. △ having AB for hypotenuse.

Ex. 133. Prove that the locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment capable of containing the given angle.

Let AB be the given base. Upon AB construct a segment capable of containing the given angle. (§ 290)



PROOF. (i.) Join any point C in the arc of this segment to A and B.

Then $\angle ACB =$ the given angle.

(ii.) Join any point D in AC to B.

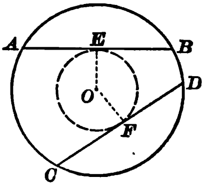
Then $\angle ADB > \angle ACB$. (§ 146)

Produce AC to any point E, and join EB.

Then $\angle AEB < \angle ACB$. (§ 146)

Therefore the arc of a segment constructed upon AB as base, and capable of containing the given angle, is the locus of the vertex of a △ having AB for base, and the given angle for the ∠ at the vertex.

Ex. 134. Find the locus of the middle points of all chords of a given length that can be drawn in a given circle.



Let O be the centre of the given ⊙, AB, CD two chords equal in length.

PROOF. Draw $OE \perp$ to AB, $OF \perp$ to CD.

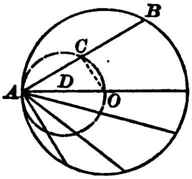
Then $AE = BE$, $CF = DF$. (§ 232)

Also $OE = OF$. (§ 236)

Similarly, the middle points of all chords equal to AB are equidistant from O.

Therefore the required locus is the circumference of a ⊙ concentric with the given ⊙, and having for radius the distance from the centre of the given ⊙ to any one of the chords.

Ex. 135. Find the locus of the middle points of all chords that can be drawn through a given point A in a given circumference.



Let O be the centre of the given ⊙, A any point in its circumference, AB any chord drawn through A, C its middle point.

Proof. Join OC.

Then $\angle ACO = 90°$. (Ex. 79)

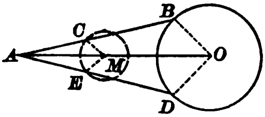
$\therefore$ C is in the circumference described upon AO as diameter. (Ex. 132)

Similarly, the middle point of every chord drawn from A lies in this circumference.

Therefore this circumference is the required locus.

Ex. 136. Find the locus of the middle points of all straight lines that can be drawn from a given exterior point A to a given circumference.

Let O be the centre of the given ⊙, AB any one of the straight lines, C its middle point.



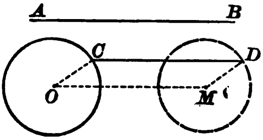
Proof. Join OA, OB, bisect AO at M, and join CM.

MC is $\parallel$ to OB, and $= \frac{1}{2} OB$. (§ 189)

Similarly, the middle point E of any other line AD may be proved to be at a distance from M equal to $\frac{1}{2} OB$.

Therefore the required locus is the circumference of a ⊙ having the middle point of AO for centre, and $\frac{1}{2} OB$ for radius.

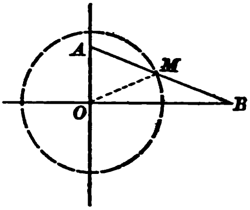
Ex. 137. A straight line moves so that it remains parallel to a given line, and touches at one end a given circumference. Find the locus of the other end.



Let O be the centre of the given ⊙, AB the given line, CD the moving line touching the given ⊙ at C and $\parallel$ to AB.

Proof. Draw OM $\parallel$ to AB and equal to CD. Join OC, MD. The figure $OCDM$ is a ▱ (§ 182). $\therefore MD = OC$ (§ 179). Therefore the distance of D from M is constant, and the locus of D is a circumference equal to the given circumference and having the point M for centre.

Ex. 138. A straight rod moves so that its ends constantly touch two fixed rods which are perpendicular to each other. Find the locus of its middle point.



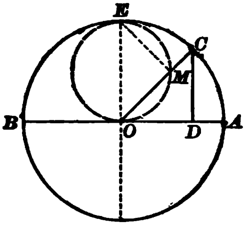
Let OA, OB be two fixed rods ⊥ to each other, AB the moving rod, M its middle point.

Proof. Join OM. Since $\angle AOB = 90°$, the point M is the centre of a ⊙ which passes through O, A, and B. (Ex. 132)

$$\therefore OM = \tfrac{1}{2} AB.$$

That is to say, OM is constant in length, and the locus is the circumference of a ⊙ having O for centre and OM for radius.

Ex. 139. In a given circle let AOB be a diameter, OC any radius, CD the perpendicular from C to AB. Upon OC take $OM = CD$. Find the locus of the point M as OC turns about O.



Proof. Draw the radius OE ⊥ to AB, and join EM.

In the ⚠ OEM, OCD,

$$OE = OC,\ OM = CD,$$

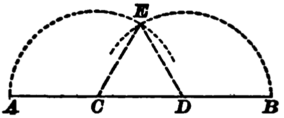
and $\angle EOM = \angle OCD$. (§ 104)

$\therefore \triangle OEM = \triangle OCD$. (§ 150)

$\therefore \angle OME = \angle ODC = 90°$.

Therefore the required locus of the point M is the circumference of a ⊙ described upon the radius OE as a diameter. (Ex. 132)

Ex. 140. To construct an equilateral triangle, having given the perimeter.



Let AB be the given perimeter.

Trisect AB at the points C, D. (§ 279)

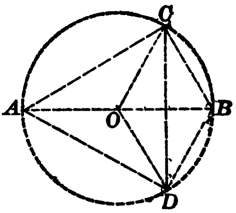
With centres C and D, and radius AC, describe arcs intersecting at E.

Proof. Join CE, DE. $\triangle CDE$ is the equilateral $\triangle$ required, as is obvious from the construction.

Ex. 141. To construct an equilateral triangle, having given the radius of the circumscribed circle.

Describe a ⊙ with the given radius. Draw any diameter AOB. With centre B and radius R cut the ⊙ at C and D.

Join AC, AD, CD. $\triangle ACD$ is the $\triangle$ required.



Proof. Join OC, OD, BC, BD.

$\triangle\ OBC$ is equilateral by construction.

$$\therefore \angle BOC = 60^\circ. \quad (§\ 155)$$

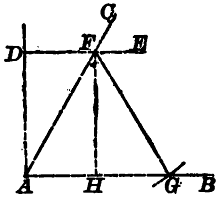
$$\therefore \angle COD = 120^\circ.$$

And $\angle AOC = \angle AOD = 120^\circ$. (§ 90)

$$\therefore \text{arc } AC = \text{arc } AD = \text{arc } DC. \quad (§\ 229)$$

$$\therefore \quad AC = AD = DC. \quad (§\ 230)$$

Ex. 142. To construct an equilateral triangle, having given the altitude.



Construct an $\angle\ BAC$ equal to 60°.

Erect at A a $\perp$ AD equal to the given altitude. Draw $DE \parallel$ to AB, meeting AC at F.

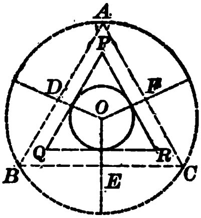
With centre F and radius FA cut AB at G. Join FG.

$\triangle\ AFG$ is the $\triangle$ required.

Proof. The altitude $FH = AD = h$. (Cons. and § 180)

The $\triangle\ AFG$ is isosceles (Cons.), and has one $\angle\ A$ equal to 60° (Cons.), it is therefore equilateral (Ex. 77).

Ex. 143. To construct an equilateral triangle, having given the radius of the inscribed circle.



With the given radius and centre O describe a ⊙; then describe a larger concentric ⊙, and in this last circle inscribe an equilateral $\triangle$ (Ex. 141).

Drop from O to the sides AB, AC, BC the $\perp$s OD, OF, OE, respectively. (§ 272)

Through the points where these $\perp$s cut the smaller ⊙ draw tangents intersecting in P, Q, R.

Then $\triangle\ PQR$ is the $\triangle$ required.

Proof. The tangent PQ is $\perp$ to OD (§ 240), therefore $\parallel$ to AB. (§ 100)

Similarly, PR is $\parallel$ to AC, and $QR \parallel$ to BC.

$$\therefore \angle QPR = \angle BAC = 60^\circ. \quad (§\ 112)$$

Similarly, $\angle PQR = \angle PRQ = 60^\circ$.

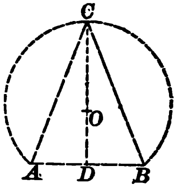
$$\therefore \triangle PQR \text{ is equilateral.} \quad (§\ 157)$$

Ex. 144. To construct an isosceles triangle, having given the base and the angle at the vertex.

Draw a line *AB* equal to the given base.

Construct upon *AB* a segment capable of containing the given ∠ *M*. (§ 290)

Draw the ⊥ bisector of *AB*, and let it cut the arc at *C*. Join *AC*, *BC*. △ *ABC* is the △ required, as is obvious from the construction.



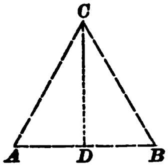
Ex. 145. To construct an isosceles triangle, having given the altitude and the angle at the vertex.

Construct ∠ *ACB* equal to the given ∠.

Draw *CD* bisecting ∠ *ACB*. (§ 275)

Take *CD* equal to the given altitude, and erect at *D* a ⊥ meeting the sides of the ∠ at *A* and *B*. (§ 271)

△ *ABC* is the △ required.



Proof. The △ *ABC* has the given parts by the construction; also, △ *ACD* = △ *BCD* (§ 149), and *AC* = *BC*.

Ex. 146. To construct an isosceles triangle, having given the base and the radius of the circumscribed circle.

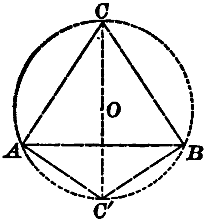
Describe a ⊙ with the given radius.

Draw a chord *AB* equal to the given base.

Construct the ⊥ bisector of *AB*, and let it meet the ⊙ at *C* and *C′*.

Join *AC*, *AC′*, *BC*, *BC′*.

▲ *ABC*, *ABC′* answer the given conditions; as is obvious from the construction and § 122.



Discussion. If the given base is greater than twice the given radius, the problem is impossible.

Ex. 147. To construct an isosceles triangle, having given the base and the radius of the inscribed circle.

Draw a line *AB* equal to the given base.

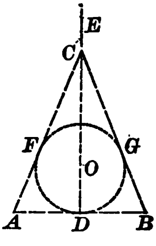
Erect the ⊥ bisector *DE*. (§ 273)

On *DE* take *DO* equal to the given radius.

With centre *O* and radius *DO* describe a ⊙.

Draw tangents to the ⊙ from *A* and *B*, and let them intersect at *C*. (§ 287)

△ *ABC* is the △ required.



Proof. Let AC met the $\odot$ at F, and BC meet it at G.

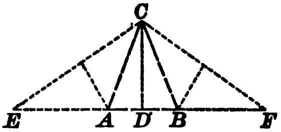
Then $AF = AD$, and $BG = BD$. (§ 246)

But $AD = BD$. $\therefore AF = BD$.

Also $FC = GC$.

Adding, we have $AC = BC$.

Ex. 148. To construct an isosceles triangle, having given the perimeter and the altitude.



Draw EF equal to the given perimeter. Bisect EF at D, and erect a $\perp$ CD equal to the given altitude.

Join EC, FC, and draw their $\perp$ bisectors. Let these bisectors cut EF at A and B. Join AC, BC. $\triangle ABC$ is the $\triangle$ required.

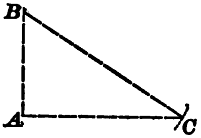
Proof. $CE = CF$, $CA = EA$, $CB = FB$. (§ 122)

$\therefore CA + AB + BC = EA + AB + BF = EF$.

Also $\angle E = \angle F = \angle ACE = \angle BCF$. (§ 154)

$\therefore \triangle ACE = \triangle BCF$ (§ 147), and $AC = BC$.

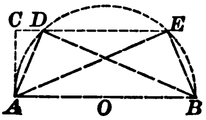
Ex. 149. To construct a right triangle, having given the hypotenuse and one leg.



Make a rt. $\angle BAC$. (§ 271)

Take AB equal to the given leg, and with centre B and radius equal to the given hypotenuse cut AC at C. $\triangle ABC$ is the $\triangle$ required.

Ex. 150. To construct a right triangle, having given the hypotenuse and the altitude upon the hypotenuse.



Upon a line AB equal to the given hypotenuse construct a semicircle. Erect at A a $\perp$ AC equal to the given altitude.

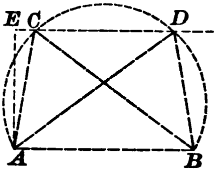
Through C draw a line $\parallel$ to AB, cutting the semicircle at D and E.

Join DA, DB, EA, EB. The $\triangle$s ADB, AEB satisfy the given conditions.

Proof. The $\angle$s at D and E are rt. $\angle$s (§ 264), and the altitudes of the $\triangle$s are equal each to AC (§ 181).

Discussion. Since the $\triangle$s ADB, AEB are equal, there is really only one solution. If $AC = \frac{1}{2} AB$, CE will be a tangent and the $\triangle$ will be isosceles; if $AC > \frac{1}{2} AB$, the problem is impossible.

Ex. 157. To construct a triangle, having given the base, the altitude, and the angle at the vertex.



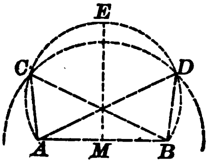
Analysis. If AB is the given base, one locus for the vertex is an arc forming with AB a segment capable of containing the given angle (§ 267). Another locus for the vertex is a parallel to AB, drawn at a distance from AB equal to the given altitude.

Construction. Upon a line AB equal to the given base, construct a segment capable of containing the given $\angle$ at the vertex (§ 290). Erect a $\perp$ AE equal to the given altitude. Draw through E a line $\parallel$ to AB, cutting the arc at C and D. Join AC, BC, AD, BD.

The $\triangle$ ABC, ABD both satisfy the given conditions, as is evident from the construction and from § 181.

Discussion. Since the $\triangle$ ABC, ABD are evidently equal, if AE equals the height of the segment there is but one $\triangle$, and the $\triangle$ is isosceles. If AE is greater than the height of the segment, the problem is impossible.

Ex. 158. To construct a triangle, having given the base, the corresponding median, and the angle at the vertex.



Analysis. If AB is the given base, one locus for the vertex is an arc forming with AB a segment capable of containing the given $\angle$. (§ 267)

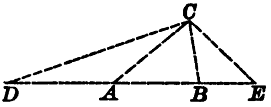
Another locus for the vertex is a circumference described with the middle point of AB for centre, and the given length of the median for radius.

Construction. Upon AB construct a segment capable of containing the given $\angle$ at the vertex. Bisect AB at M. With centre M and radius equal to the length of the given median, describe a $\odot$ cutting the arc of the segment at C and D. Join AC, BC, AD, BD. The $\triangle$ ABC, ABD satisfy the given conditions.

Discussion. The two $\triangle$ ABC, ABD are equal and form but one solution. In order that the problem may be possible, the given median must have a value between half the given base and the height ME of the segment constructed upon the base.

Ex. 159. To construct a triangle, having given the perimeter and the angles.

Analysis. Let ABC be the $\triangle$ required. Upon AB produced take $AD = AC$, and $BE = BC$. Then DE is equal to the given perimeter.



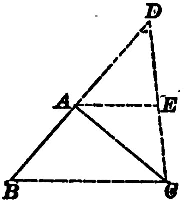
Since $\triangle$s ACD, BCE are isosceles, and $\angle$s A and B of $\triangle ABC$ are exterior $\angle$s, it is easily seen that $\angle ADC = \frac{1}{2} \angle BAC$, and $\angle BEC = \frac{1}{2} \angle ABC$ (§§ 154, 145). Hence the following construction:

Construction. Draw DE equal to the given perimeter. Construct at D an $\angle$ equal to $\frac{1}{2} \angle A$, and at E an $\angle$ equal to $\frac{1}{2} \angle B$; this determines the point C. Draw through C a line CA, making $\angle ACD$ equal to $\frac{1}{2} \angle A$, and a line CB, making $\angle BCE$ equal to $\frac{1}{2} \angle A$.

$\triangle ABC$ is the $\triangle$ required.

Ex. 160. To construct a triangle, having given one side, an adjacent angle, and the sum of the other two sides.

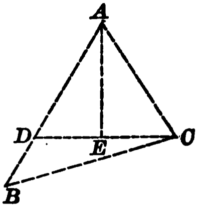
Analysis. Suppose the problem solved, and ABC to be the $\triangle$ required. We know BC, the $\angle B$, and the sum $BA + AC$. Produce BA, making $AD = AC$, and join DC. $\triangle ADC$ is isosceles. We now see that $\triangle BDC$ is determined; for $\angle B$ and BC are given, and we can make $BD = BA + AC$. After $\triangle BDC$ has been constructed, the point A may be found by erecting a $\perp$ at the middle point of DC.



Construction. Construct $\triangle BDC$ (§ 280). Draw the $\perp$ bisector of DC, cutting BD at A, and join AC. $\triangle ABC$ is the $\triangle$ required.

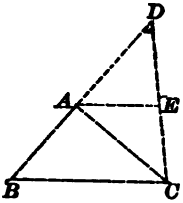
Ex. 161. To construct a triangle, having given one side, an adjacent angle, and the difference of the other two sides.

Analysis. Suppose the problem solved, and let ABC be the $\triangle$ required. We know BC, the $\angle B$, and the difference $AB - AC$. If, then, we take upon AB a length $BD = AB - AC$, we obtain a $\triangle BDC$, which we can construct from the given parts; then a $\perp$ erected at the middle point of DC will meet BD produced, and determine the point A.



Construction. Construct $\triangle BDC$ (§ 280), and then draw the $\perp$-bisector, meeting BD produced at A. Join AC. $\triangle ABC$ is the $\triangle$ required.

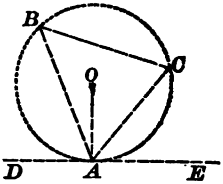
Ex. 162. To construct a triangle, having given the sum of two sides and the angles.



ANALYSIS. Let ABC be the $\triangle$ required. We know the $\angle$s and the sum $AB+AC$. Produce BA, making $AD=AC$. The $\triangle ADC$ is isosceles. $\therefore \angle D = \frac{1}{2}\angle BAC$ (§§ 145, 154); since also $\angle B$ and BD are known, $\triangle BDC$ is determined. Hence the construction.

CONSTRUCTION. Construct $\triangle BDC$ (§ 281), and draw the $\perp$ bisector of DC, meeting BD at A. Join AC. $\triangle ABC$ is the $\triangle$ required.

Ex. 163. To construct a triangle, having given one side, an adjacent angle, and the radius of the circumscribed circle.



With the given radius describe a $\odot$. In this $\odot$ draw a chord AB equal to the given side. Through A draw another chord AC, making the $\angle BAC$ equal to the given $\angle$. Join BC.

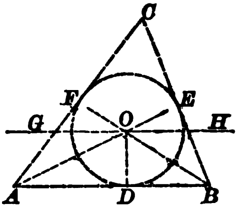
$\triangle ABC$ is the $\triangle$ required.

Ex. 164. To construct a triangle, having given the angles and the radius of the circumscribed circle. (Fig. of Ex. 163.)

With the given radius describe a $\odot$. Through any point A of the circumference draw a tangent DAE (§ 287). Through A draw the chords AB, AC, making the $\angle$s DAB, EAC equal respectively to two of the given $\angle$s. Join BC. $\triangle ABC$ is the $\triangle$ required.

PROOF. $\angle ABC = \angle EAC$, because each $\angle$ is measured by half the arc AC (§§ 263, 269). Similarly, $\angle ACB = \angle DAB$.

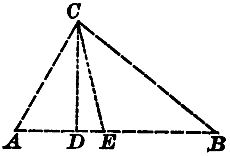
Ex. 165. To construct a triangle, having given the angles and the radius of the inscribed circle.



ANALYSIS. Let ABC be the $\triangle$ required, OD the radius of the inscribed $\odot$, meeting AB at D. We know the $\angle$s A, B, C, and the length OD. We also know that O is the intersection of the bisectors of the $\angle$s A, B, C; also that its distance from AB is equal to OD. Finally, we see that if GH is a line drawn through O $\parallel$ to AB, then $\angle BOH = \angle OBD = \frac{1}{2}\angle ABC$.

CONSTRUCTION. Construct the $\angle A$ and its bisector. Draw GH $\parallel$ to AB at the given distance OD from AB (§§ 271, 278), and let GH meet the bisector at O. Draw OB, meeting AB at B, and making $\angle BOH$ equal to one-half the given $\angle B$. Draw BC, meeting AC at C, and making $\angle OBC$ equal to one-half the given $\angle B$. $\triangle ABC$ is the $\triangle$ required.

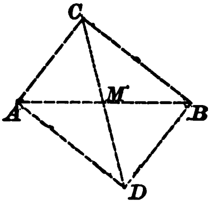
Ex. 166. To construct a triangle, having given an angle, the bisector, and the altitude drawn from the vertex.



Analysis. Let ABC be the $\triangle$ required There are given $\angle C$, the altitude CD, and the bisector CE. Therefore the rt. $\triangle CDE$ is determined, and also the ∡ ECA, ECB, each being one-half of the given $\angle C$.

Construction. Construct $\triangle CDE$ (Ex. 149). Draw ECA and ECB, each equal to one-half the given $\angle C$ (§ 276). Produce DE in both directions. $\triangle ABC$ is the $\triangle$ required.

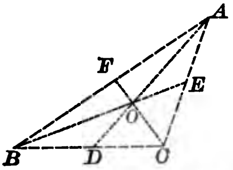
Ex. 167. To construct a triangle, having given two sides and the median corresponding to the third side.



Analysis. Let ABC be the $\triangle$ required. We know AC, BC, and the median CM. Produce CM, making $MD = CM$, and join DA, DB. The Δ AMD, BMC are equal (§ 150), hence $AD = BC$. Therefore the $\triangle ACD$ is determined, because its sides are all known; and when it has been constructed, the point B is found by producing AM so that $MB = AM$.

Construction. Construct $\triangle ACD$ (§ 282), draw AM, and produce it to B, making $MB = AM$. Join BC. $\triangle ABC$ is the $\triangle$ required.

Ex. 168. To construct a triangle, having given the three medians.

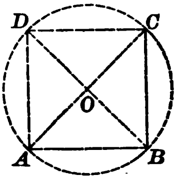


Analysis. Let ABC be the $\triangle$ required, AD, BE, CF the medians, O their common point of intersection (Ex. 33). We know that the distance from O to each vertex is two-thirds the length of the corresponding median. Therefore the $\triangle BOC$ is determined, since we know OB, OC, and OD (Ex. 167). By producing BO by a length $OE = \frac{1}{2} BO$, and CO by a length $OF = \frac{1}{2} OC$, we obtain the points E and F. Finally, the lines BF, CE produced will meet at the point A.

Construction. Construct the $\triangle BOC$ (Ex. 167). Produce BO, making $OE = \frac{1}{2} BO$, and produce CO, making $OF = \frac{1}{2} OC$. Draw BF and CE, and produce them to meet at A. $\triangle ABC$ is the $\triangle$ required.

Ex. 169. To construct a square, having given the diagonal.

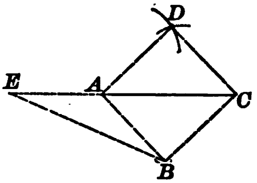
ANALYSIS. Suppose the problem solved, $ABCD$ the square, AC the diagonal. We know AC, and also that the ∠s at B and D are rt. ∠s. Therefore B and D must lie in the circumference of a ⊙ having AC for diameter. Also, since B and D are each equidistant from A and C, B and D lie in the ⊥ bisector of AC.



CONSTRUCTION. Bisect AC at O (§ 273). With centre O and radius OA describe a ⊙, intersecting the ⊥ bisector of AC at B and D. Join AB, BC, CD, DA. $ABCD$ is the required square.

Ex. 170. To construct a square, having given the sum of the diagonal and one side.

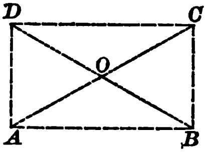
ANALYSIS. Let $ABCD$ be the square required, AC the diagonal. Produce CA to E, making $AE = AB$, and join BE; △s ABC and ABE are isosceles, and we also know that $\angle ACB = \angle CAB = 45°$; whence we can find $\angle AEB$ (§ 145); this determines $\triangle EBC$; then D can be found as in Ex. 169.



CONSTRUCTION. Draw a line EC equal to the given sum of AB and AC. Draw CB, making $\angle BCE = 45°$, and EB, making $\angle BEC = 22\frac{1}{2}°$, meeting CB at B. Erect BA ⊥ to BC. With centres A and C, and radius AB, describe arcs intersecting at D. Join DA, DC. $ABCD$ is the square required.

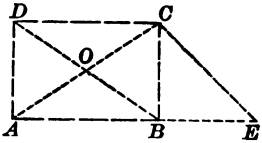
Ex. 171. To construct a rectangle, having given one side and the angle formed by the diagonals.

ANALYSIS. Let $ABCD$ be the rectangle required, AB the given side, $\angle AOB$ the given ∠. Since the diagonals of a rectangle are equal (Ex. 26), and bisect each other (§ 184), we know that $AO = BO$; hence the isos. $\triangle AOB$ is determined, and may be constructed. The points C and D are then found by properly producing AO and BO.



CONSTRUCTION. Construct the $\triangle AOB$ (Ex. 144). Produce AO, making $OC = AO$, and BO, making $OD = BO$. Join BC, CD, DA. $ABCD$ is the rectangle required (Ex. 25).

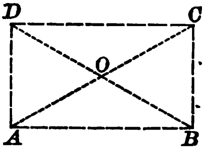
Ex. 172. To construct a rectangle, having given the perimeter and the diagonal.



Analysis. Let $ABCD$ be the rectangle. In the rt. $\triangle ABC$ we know AC and $AB+BC$. To construct this $\triangle$, prolong AB to E, making $BE=BC$. At E make an $\angle$ of 45°, and let its side meet at C the ⊙ described from A as centre and with AC as radius. From C drop the $\perp$ CB.

Construction. Construct the rt. $\triangle ABC$. Bisect AC at O. Join BO, and produce it to D, making $OD=BO$. Join DC and DA.

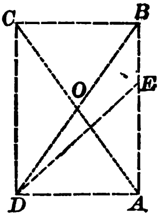
Ex. 173. To construct a rectangle, having given the perimeter and the angle of the diagonals.



Analysis. Let $ABCD$ be the rectangle required, $\angle AOB$ the given $\angle$. Since $\triangle OAB$ is isosceles, the $\angle OAB$ may be easily found, and then its complement, the $\angle ACB$. Hence, in the rt. $\triangle ABC$, we know the $\angle$s and the sum $AB+BC$ (which is equal to half the given perimeter). Therefore we construct the $\triangle ABC$, and then proceed as in Ex. 172.

Construction. Make $\angle OAB = \frac{1}{2}(180° - \angle AOB)$. Construct the $\triangle$ ABC (Ex. 162). Bisect AC at O. Join BO and produce it, making $OD=OB$. Join DC, DA. $ABCD$ is the rectangle required (Ex. 25).

Ex. 174. To construct a rectangle, having given the difference of two sides and the angle of the diagonals.



Analysis. Let $ABCD$ be the rectangle required, and E a point in AB, such that $AE = AD$. We know EB, the $\angle AOB$, and that $\angle B = \frac{1}{2}(180° - \angle AOB)$. In $\triangle ADE$,

$$AE = AD;\ \therefore \angle AED = \angle ADE;$$

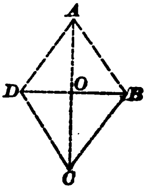
and since the $\triangle$ is a rt. $\triangle$,

$\therefore \angle AED = 45°$, and $\angle BED = 180° - 45° = 135°$.

Therefore in $\triangle BDE$, a side and the adjacent $\angle$s are known. After this $\triangle$ has been constructed, drop a $\perp$ from D to meet BE produced, etc.

Construction. Construct the $\triangle$ BDE (§ 281). From D draw $DA \perp$ to BE, meeting BE produced at A. Bisect BD at O. Join AO, and produce it, making $OC=AO$. Join CB, CD. $ABCD$ is the rectangle required (Ex. 25).

Ex. 175. To construct a rhombus, having given the two diagonals.

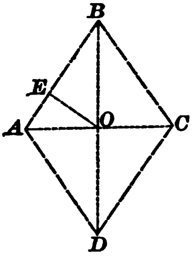


Analysis. The diagonals of a rhombus bisect each other (§ 184), and are ⊥ to each other (Ex. 28): hence the construction is obvious.

Construction. Draw AC equal to one diagonal. Draw the ⊥ bisector of AC, cutting AC at O; upon it lay off in opposite directions from C the lengths OB, OD, and equal to half the other diagonal.

$ABCD$ is the rhombus required.

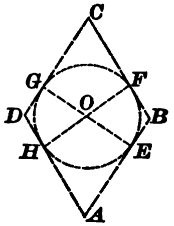
Ex. 176. To construct a rhombus, having given one side and the radius of the inscribed circle.



Analysis. Since the diagonals of a rhombus bisect each other (Ex. 28), their intersection O is a point equidistant from the sides (§ 164), and therefore the centre of the inscribed ⊙. Hence, if we draw OE ⊥ to AB, OE will be equal to the radius of the inscribed ⊙. Now in the rt. △ ABO we know the hypotenuse AB and the altitude OE. This △ once constructed, the rest is obvious.

Construction. Construct the △ AOB (Ex. 150). Produce AO, making $OC = AO$, and BO, making $OD = BO$. Join AD, DC, CB. $ABCD$ is the rhombus required.

Ex. 177. To construct a rhombus, having given one angle and the radius of the inscribed circle.



Analysis. Let $ABCD$ be the rhombus required, $\angle ABC$ the given $\angle$, O the centre of the inscribed ⊙. Draw radii to the points of contact E, F, G, H.

The $\angle FOG = 180° - \angle GCF$, (§§ 206, 240)

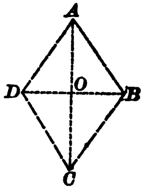
and $\angle ABC = 180° - \angle GCF$. (§ 109)

$\therefore \angle FOG = \angle ABC$.

Since also $\angle EOF = 180° - \angle ABC$, GOE is a straight line (§ 93). Similarly, FOH is a straight line. Hence we obtain the following construction:

Construction. With centre O and the given radius, describe a ⊙. Draw through O two diameters GOE, FOH, making with each other the $\angle GOF$ equal to the given $\angle$. Through the points E, F, G, H, draw tangents intersecting in A, B, C, D. $ABCD$ is the required rhombus.

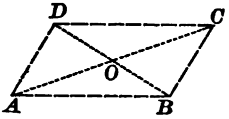
Ex. 178. To construct a rhombus, having given one angle and one of the diagonals.



ANALYSIS. Let $ABCD$ be the rhombus required, $\angle ABC$ the given $\angle$, AC the given diagonal. The $\angle BAD = 180° - \angle ABC$ (§ 109), and is therefore known. Also the diagonals of a rhombus bisect each other at rt. $\angle$s, and bisect the $\angle$s of the rhombus. (Ex. 28)

CONSTRUCTION. Draw AC. Through A draw AB and AD, each making an $\angle$ with AC equal to $\frac{1}{2}(180° - \angle ABC)$ Draw the $\perp$ bisector of AC, meeting AB at B and AD at D. Join CB, CD. The figure $ABCD$ is the rhombus required.

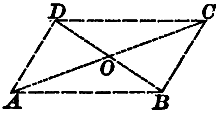
Ex. 179. To construct a rhomboid, having given one side and the two diagonals.



ANALYSIS. Let $ABCD$ be the rhomboid required, AB the given side, O the intersection of the diagonals. Since the diagonals bisect each other (§ 184) we know the three sides of the $\triangle AOB$; then C and D are found by producing AO and BO.

CONSTRUCTION. Construct the $\triangle AOB$ (§ 282). Produce AO, making $OC = AO$, and BO, making $OD = BO$. Join BC, AD, and DC. $ABCD$ is the figure required.

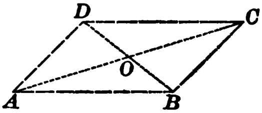
Ex. 180. To construct a rhomboid, having given the diagonals and the angle formed by them.



ANALYSIS. Let $ABCD$ be the rhomboid required, $\angle AOB$ the given $\angle$, O the intersection of the diagonals. In the $\triangle AOB$ we know two sides OA, OB and the included $\angle$. (§ 184),

CONSTRUCTION. Construct the $\triangle AOB$, and then proceed as in Ex. 179.

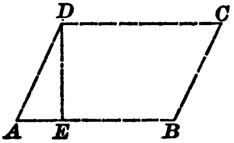
Ex. 181. To construct a rhomboid, having given one side, one angle, and one diagonal.



ANALYSIS. Let $ABCD$ be the rhomboid, $\angle A$ the given $\angle$, AB the given side, AC the given diagonal. The $\angle ABC = 180° - \angle BAD$. Hence the $\triangle ABC$ is determined.

CONSTRUCTION. Construct the $\triangle ABC$ (§ 283). Through A draw $AD \parallel$ to BC, and through C draw $CD \parallel$ to BA. $ABCD$ is the figure required.

Ex. 182. To construct a rhomboid, having given the base, the altitude, and one angle.

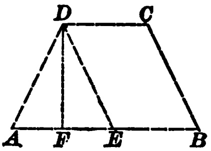


Analysis. Let *ABCD* be the rhomboid required. There is given *AB*, the $\angle A$, and the altitude *DE*. Therefore the $\angle ADE$ is known, and the rt. $\triangle ADE$ is determined.

Construction. Construct the rt. $\triangle ADE$ (§ 281). Produce *AE*, making *AB* equal to the given base. Through *B* draw $BC \parallel$ to *AD*, and through *D* draw $DC \parallel$ to *AB*.

The figure *ABCD* is the rhomboid required.

Ex. 183. To construct an isosceles trapezoid, having given the bases and one angle.



Analysis. Let *ABCD* be the trapezoid required, $\angle A$ the given $\angle$.

Draw $DE \parallel$ to *CB*.

$DE = BC.$ (§ 180)

But $BC = AD.$

$\therefore DE = AD.$ $\therefore \angle DEA = \angle A.$

Also, since $AE = AB - EB$ and $EB = DC$, $AE = AB - DC$, and is therefore known. $\therefore$ the $\triangle ADE$ is determined. This $\triangle$ constructed, the rest of the construction is obvious.

Construction. Construct the $\triangle ADE$ (§ 281). Produce *AE*, making *AB* equal to the longer base. Draw *DC* equal to the shortêr base and $\parallel$ to *AB*. Join *BC*. The figure *ABCD* is the figure required.

Ex. 184. To construct an isosceles trapezoid, having given the bases and the altitude.

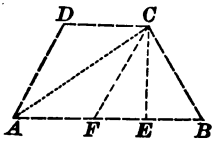
Analysis. Let *ABCD* (see figure to last exercise) be the trapezoid required, *DF* the altitude. Draw $DE \parallel$ to *CB*. The $\triangle ADE$ is isosceles. (see last Analysis), and $AE = AB - DC$. Also *DF* bisects *AE* (Ex. 16).

Construction. Draw *AB* equal to the longer base. Upon *AB* lay off $AE = AB - DC$. Draw the $\perp$ bisector to *AE* and lay off upon it a length *FD* equal to the given altitude. Join *AD*. Through *B* draw $BC \parallel$ to *ED*, and through *D* draw $DC \parallel$ to *AB*.

The figure *ABCD* is the figure required.

Ex. 185. To construct an isosceles trapezoid, having given the bases and one diagonal.

Analysis. Let $ABCD$ be the trapezoid required, AC the given diagonal. Draw the altitude CE, and also draw $CF \parallel$ to DA. Then $AF = DC$ (§ 180), and $FB = AB - DC$. Also, since the $\triangle BCF$ is isosceles (see Analysis to Ex. 183), $FE = \frac{1}{2} FB$ (Ex. 16).
$\therefore AE = DC + \frac{1}{2}(AB - DC)$, and is known.
$\therefore$ the rt. $\triangle ACE$ can be constructed. When this has been done, the rest is obvious.



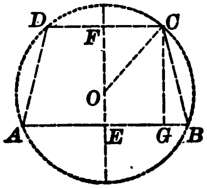
Construction. Construct the rt. $\triangle ACE$ (Ex. 149). Produce AE, making AB equal to the longer base. Join BC. Draw $DC \parallel$ to AB and equal to the shorter base. Join AD.

The figure $ABCD$ is the trapezoid required.

Ex. 186. To construct an isosceles trapezoid, having given the bases and the radius of the circumscribed circle.

Analysis. Let $ABCD$ be the trapezoid required, O the centre of the circumscribed ⊙. A diameter drawn ⊥ to AB will also be ⊥ to CD (§ 102), and will bisect both AB and CD (§ 232). Let this diameter meet AB at E, CD at F. Draw $CG \parallel$ to FE, meeting AB at G; then $EG = FC = \frac{1}{2} DC$.

Hence we have the following construction:

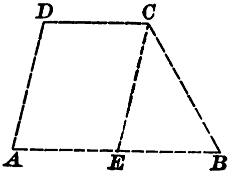


Construction. With any centre O and the given radius describe a ⊙. Place in this ⊙ a chord AB equal to the longer base. Draw a diameter ⊥ to AB, meeting AB at E. Upon AB lay off $EG = \frac{1}{2}$ the shorter base. Draw $GC \parallel$ to EO, meeting the ⊙ at C. Draw $CD \parallel$ to BA, meeting the ⊙ again at D. Join BC, AD.

$ABCD$ is the trapezoid required.

Ex. 187. To construct a trapezoid, having given the four sides.

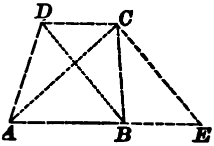
Analysis. Let $ABCD$ be the trapezoid required. Draw $CE \parallel$ to AD. $\triangle CBE$ is determined, because BC is given, $CE = AD$ (§ 179), and $EB = AB - CD$.



Construction. Construct $\triangle BCE$ (§ 282). Produce BE to A, making AB equal to the longer base. Draw $AD \parallel$ to EC and equal to EC, and join DC.

$ABCD$ is the required trapezoid.

Ex. 188. To construct a trapezoid, having given the bases and the diagonals.



Analysis. Let $ABCD$ be the trapezoid required. Draw CE ∥ to the diagonal DB, and produce AB to meet CE at E. $\triangle AEC$ is determined, since AC is given, $CE = BD$ (which is given), and $AE = AB + DC$.

Construction. Construct $\triangle AEC$ (§ 282). Through B draw BD ∥ and equal to EC. Join AD, DC, BC.

The figure $ABCD$ is the figure required.

Ex. 189. To construct a trapezoid, having given the bases, one diagonal, and the angle formed by the diagonals.

Analysis. (See Fig. in Ex. 188.) Let AB, DC be the given bases, AC the given diagonal. The $\angle ACE$ equals the angle of the diagonals. If this angle is acute, and AE is greater than, or equal to, AC, the $\triangle ACE$ is determined (§ 283).

Construction. Construct $\triangle AEC$. Upon AE take AB equal to one of the bases. Draw CD ∥ to EA and equal to the other base. Join BC and AD. The figure $ABCD$ is the trapezoid required.

Discussion. For other cases, see the discussion of the problem in § 283.

Ex. 190. Find the locus of the centre of a circle which has a given radius r and passes through a given point P.

The required locus is the circumference of a ⊙ having P for centre and r for radius.

Ex. 191. Find the locus of the centre of a circle which has a given radius r and touches a given straight line AB.

The required locus consists of the two straight lines drawn ∥ to AB, one on one side of AB, the other on the other side, at the distance r from AB.

Ex. 192. Find the locus of the centre of a circle which passes through two given points P and Q.

The locus is the ⊥ bisector of the line joining the given points (§ 233).

Ex. 193. Find the locus of the centre of a circle which touches a given straight line AB at a given point P.

The required locus is the ⊥ to AB erected at P (§ 241).

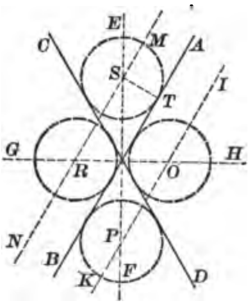
Ex. 194. Find the locus of the centre of a circle which touches each of two given parallels.

This locus is the straight line ∥ to the given parallels and mid-way between them.

Ex. 195. Find the locus of the centre of a circle which touches each of two given intersecting lines.

The locus consists of the two bisectors of the ∠s formed by the given lines (§ 164).

Ex. 196. To construct a circle, having a given radius r and touching each of two intersecting lines AB and CD.

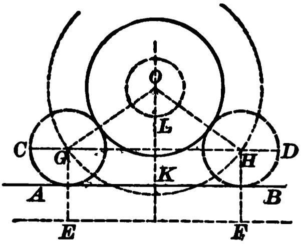


Analysis. One locus of the centre consists of the bisectors of the ∠s formed by the lines AB and CD (Ex. 195). Another locus consists of the two parallels to either AB or CD, drawn at a distance from it equal to the given radius.

Construction. Draw the bisectors EF, GH of the ∠s formed by the given lines AB, CD. Draw IK and MN ∥ to AB, and at a distance from AB equal to the given radius r. Let the parallels cut the bisectors at the points O, P, R, S. With these points as centres and r as radius, describe Ⓢ: each ⊙ will satisfy the given conditions.

Ex. 197. To construct a circle, having a given radius r touching a given line AB and touching a given circle.

Analysis. Let O be the centre of the given ⊙, R its radius. One locus of the centre of the required ⊙ consists of the parallels to AB drawn at the distance r from AB. Another locus consists of the circumferences described about O as centre with the radii $R + r$ and $R - r$. The points in which the two parallels intersect the two circumferences are the centres of Ⓢ which will satisfy the given conditions.



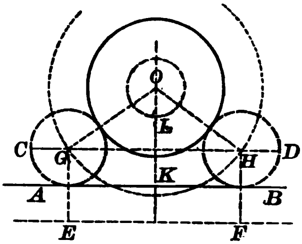
The construction of two of these Ⓢ is sufficiently evident from the figure.

Discussion. Let d denote the distance OK from O to AB.

(i.) If $d < R + 2r$, the problem is impossible.

(ii.) If $d = R + 2r$, there is *one* solution, and the centre of the ⊙ lies in the line OK.

(iii.) If d has any value between $R + 2r$ and R, there are *two* solutions, as shown in the figure.



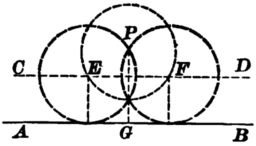
(iv.) If $d = R$, CD will pass through the point L and determine three centres, one at L, the others on the ⊙ OG. But the other parallel EF will now be tangent to the ⊙ OG and determine by the point of contact a fourth centre. Therefore in this case there are *four* solutions.

(v.) If $d < R$ but $> R - 2r$, the parallel CD will cut both auxiliary Ⓢ and determine four centres; and the parallel EF will cut the ⊙ OG and determine two centres. Therefore there will be *six* solutions.

(vi.) If $d = R - 2r$ (R being supposed $> 2r$), EF will pass through L and determine three centres; CD will determine four more. Therefore there will be *seven* solutions.

(vii.) If $d < R - 2r$, CD and EF will cut each auxiliary ⊙ twice; hence we shall have *eight* solutions.

Ex. 198. To construct a circle having a given radius r, passing through a given point P, and touching a given line AB.

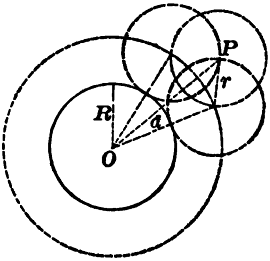


ANALYSIS. The centre of the required ⊙ must lie in the circumference described from P as centre with r as radius. It must also lie in a parallel to AB drawn at the distance r from AB on the *same* side of AB as the point P. The points common to this circumference and this parallel are the centres of Ⓢ which if described with the radius r will satisfy the given conditions.

DISCUSSION. It is obvious that the parallel to AB at the distance r on *the other side* of AB is not in general a locus for the centre of the required ⊙. Let d denote the distance PG from P to AB. If $d > 2r$, the problem is impossible. If $d = 2r$, CD will be tangent to the auxiliary ⊙, and there will be *one* solution. If $d < 2r$, there are *two* solutions (as seen in the figure). If $d = 0$, P lies in AB, and there will be two Ⓢ, both touching AB at P, one touching the parallel CD, the other the corresponding parallel on the other side of AB.

Ex. 199. To construct a circle having a given radius r, passing through a given point P, and touching a given circle.

Aɴᴀʟʏsɪs. Let R denote the radius of the given ⊙, and O denote its centre. One locus of the required centre is a circumference described from P as centre with radius r. Further, since the required ⊙ must touch the given ⊙, and the distance between the centres of two Ⓢ which touch each other is equal to the sum or the difference of their radii (Ex. 78), it follows that a second locus of the required centre is a circumference described from O as centre with a radius equal to $R+r$ or $R-r$. The points common to these loci are the centres of the Ⓢ which satisfy the given conditions.



Dɪsᴄᴜssɪᴏɴ. Let d denote the distance from P to O. In order that the ⊙ with radius $R+r$ may cut the auxiliary ⊙ described about P, it is necessary that

$$d<(R+r)+r, \text{ or } d<R+2r,$$

and $$d>(R+r)-r, \text{ or } d>R.$$

The second condition requires that P should lie without the given ⊙. Then if $d<R+2r$, there are *two* solutions; if $d=R+2r$, *one* solution; if $d>R+2r$, *no* solution.

In order that the ⊙ with radius $R-r$ may cut the auxiliary ⊙ described about P, it is necessary that

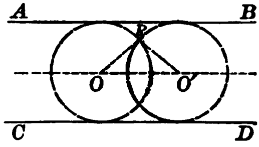
$$d<(R-r)+r, \text{ or } d<R,$$

and $$d>(R-r)-r, \text{ or } d>R-2r.$$

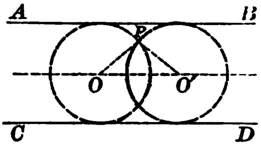
The first condition requires P to be within the given ⊙. Then, if $d>R-2r$, there are *two* solutions; if $d=R-2r$, *one* solution; if $d<R-2r$, *no* solution.

If $d=R$, P lies in the given circumference, and two Ⓢ can be described with radius r, both touching the given ⊙ at P, one externally, the other internally.

Ex. 200. To construct a circle which shall touch two given parallels and pass through a given point P.

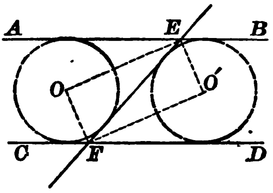


Aɴᴀʟʏsɪs. Let AB and CD be the given parallels. One locus of the required centre is the parallel drawn midway between AB and CD (Ex. 23). The radius of the required ⊙ is equal to half the distance between AB and CD. Therefore another locus for the centre is a circumference described from *P* as centre with half the distance between *AB* and *CD* for radius.



Discussion. If *P* is anywhere between *AB* and *CD*, there are *two* solutions. If *P* is on *AB* or *CD*, there is *one* solution. If *P* is outside the parallels, the problem is impossible.

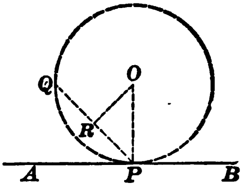
Ex. 201. To construct a circle which shall touch three given lines two of which are parallel.



Analysis. Let *AB* and *CD* be the parallel lines, and let the third line cut *AB* at *E*, and *CD* at *F*. The centres of Ⓢ that will satisfy the conditions are the intersections of the bisectors of the ∡ formed at the points *E* and *F* (Ex. 195), and must also be situated between *AB* and *CD* (Ex. 194).

Discussion. There are *two* solutions, one on each side of *EF*. It is easy to see that the four bisectors of the ∡ between the parallels *AB* and *CD* form a rectangle.

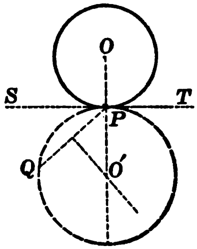
Ex. 202. To construct a circle which shall touch a given line *AB* at *P*, and pass through a given point *Q*.



Analysis. One locus for the centre is a ⊥ to *AB* erected at *P* (Ex. 193), and another locus is the ⊥ bisector of the line joining *P* and *Q* (Ex. 192).

Discussion. If *Q* is in *AB*, the problem is impossible.

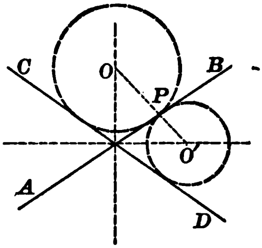
Ex. 203. To construct a circle which shall touch a given circle at *P* and pass through a given point *Q*.



Analysis. One locus for the centre is the ⊥ bisector of the line joining *P* and *Q* (Ex. 192); another locus is the straight line passing through *P* and the centre of the given ⊙ (§§ 214, 250).

Discussion. Since two straight lines intersect in only one point, there cannot be more than *one* solution. Let *ST* be the tangent to the given ⊙ drawn through the point *P*. If *Q* and the given ⊙ lie on opposite sides of *ST*, the required ⊙ will touch the given ⊙ externally. If *Q* and the given ⊙ lie on the same side of *ST*, there is internal contact; and the required ⊙ will contain the given ⊙, or *vice versa*, according as *Q* lies without or within the given ⊙. If *Q* lies on *ST*, the problem is impossible.

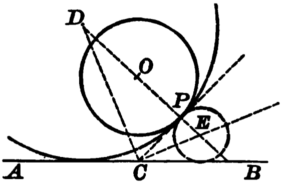
Ex. 204. To construct a ⊙ which shall touch two given lines, one of them at a given point *P*.



Analysis. Let *AB*, *CD* be the given lines, and let *P* be a point in *AB*. The centre of the required ⊙ must lie in one of the bisectors of the ∠s formed by *AB* and *CD*; and also in a ⊥ to *AB* erected at *P*.

Discussion. There are in general *two* Ⓢ satisfying the given conditions. If *P* coincides with the intersection of *AB* and *CD*, the problem is impossible. If *AB* is ∥ to *CD*, there is only *one* ⊙, and its radius is equal to half the distance between *AB* and *CD*.

Ex. 205. To construct a circle which shall touch a given line and touch a given circle at a point *P*.

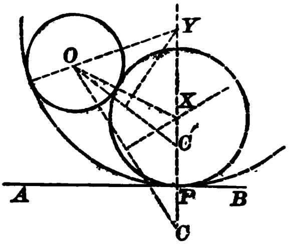


Analysis. Draw a tangent to the given ⊙ at the point *P*; then the required ⊙ must touch this tangent at *P* (§ 214). This reduces the problem to Ex. 204.

Construction. Let *AB* be the given line, *O* the centre of the given ⊙. Draw a straight line through *O* and *P*. At *P* erect a ⊥ to this line, meeting *AB* at *C*; then this ⊥ is a tangent to the given ⊙ (§ 239). Draw the bisectors of the ∠s *ACP*, *BCP*, and let them meet the line *OP* at the points *D* and *E*. With centre *D* and radius *DP* describe a ⊙; with centre *E* and radius *EP* describe a ⊙: these Ⓢ satisfy the given conditions.

Discussion. If *AB* is tangent to the given ⊙, one of the two solutions is a ⊙ coinciding with the given ⊙; if *AB* is tangent at *P*, there are an indefinite number of solutions; if *AB* cuts the given ⊙, one solution will be a ⊙ touching the given ⊙ internally; if *AB* cuts the given ⊙, and also passes through *P*, the problem is impossible.

Ex. 206. To construct a circle which shall touch a given line AB at P and also touch a given circle.



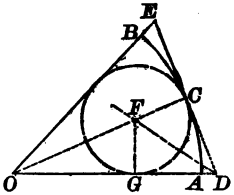
ANALYSIS. Suppose the problem solved, and let O be the centre of the given ⊙, X the centre of the required ⊙. One locus for X is the ⊥ to AB erected at P. Produce this ⊥, making $PC =$ the radius of the given ⊙; then $OX = CX$. Therefore another locus for X is the ⊥ bisector of the line joining O and C. The intersection of the two loci determines X.

DISCUSSION. Another solution is obtained by laying off $PC' =$ the given radius upon the ⊥ to AB and *on the same side* of AB as the given ⊙, joining OC', and erecting the ⊥ bisector of OC'. The ⊙ obtained by using C as the auxiliary point touches the given ⊙ externally; while the ⊙ obtained by using C' is touched by the given ⊙ internally. If AB be supposed to move ∥ to itself towards the given ⊙, then the ⊙ obtained by the use of C' will become larger and larger; at the limit when AB touches the given ⊙, it is infinitely large. If AB cuts the given circumference, the ⊙ obtained by use of C' will lie on the other side of AB, and will touch the given ⊙ externally. If AB cuts the given ⊙, and P lies within it, then both the Ⓢ will touch the given ⊙ internally.

Ex. 207. To inscribe a circle in a given sector (in other words: to construct within a given sector a circle which shall touch the arc of the sector and the two radii).

ANALYSIS. Let AOB be the given sector, and let the bisector of the ∠ AOB meet the arc AB at C. We know that the centre of the ⊙ required must lie in OC (§ 164).

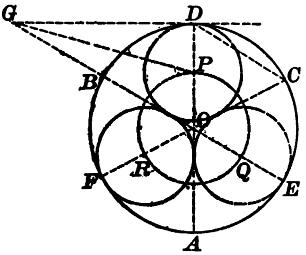
Draw a tangent to the given arc through C, and let it meet OA produced at D and OB produced at E; then we also know that the required ⊙ will touch this tangent (§ 214).



CONSTRUCTION. Draw OC, the bisector of the ∠ AOB (§ 275). Through C draw a tangent meeting OA produced at D and OB produced at E. Bisect the ∠ ODE, and let the bisector meet OC at F. With centre F and radius FC describe a ⊙. This ⊙ will be the ⊙ required.

Ex. 208. To construct within a given circle three equal circles. so that each shall touch the other two and also the given circle.

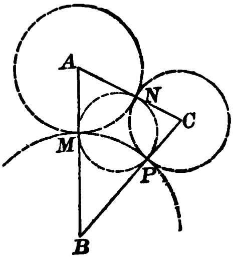
Analysis. Suppose the problem solved, and let O be the centre of the given ⊙, and P, Q, R the centres of the required Ⓢ. From the equality of the Ⓢ it is clear that their centres must be symmetrically arranged around O; and hence we see that the ∠s at O formed by OP, OQ, OR, are each 120°. We also see that these ∠s are bisected by the radii OA, OB, OC, drawn tangent to the three Ⓢ (§ 246); and therefore that OA, OB, OC, also form at O ∠s each equal to 120°. The problem is now reduced to inscribing Ⓢ in sectors of 120° (Ex. 207). In order to construct the sectors of 120° each, we produce one radius AO to D, join DC, and observe that, since $\triangle ODC$ is isosceles and $\angle DOC = 60°$, therefore the $\triangle$ is equilateral, and $DC = DO$ (Ex. 77).



Construction. Draw a diameter AD. With centre D and radius $DO = \frac{1}{2}AD$, cut the circumference at B and C. Draw the diameters BOE, COF. Draw a tangent at D (§ 287), and produce OB to meet it at G. Bisect $\angle OGD$ (§ 275), and let the bisector meet OD at P. With centre O and radius OP, cut BE at Q, CF at R. With P, Q, R as centres, and radius PO, describe Ⓢ. These Ⓢ will be the Ⓢ required.

Ex. 209. To describe circles about the vertices of a given triangle as centres, so that each shall touch the two others.

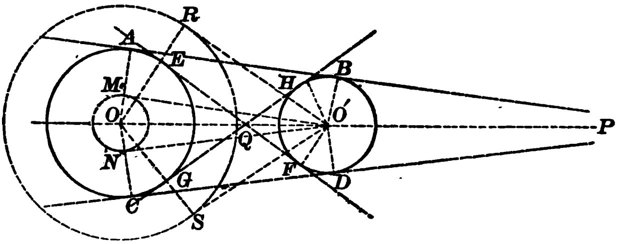
Analysis. Suppose the problem solved, ABC the required $\triangle$, and let the Ⓢ touch one another at M, N, P. Since $AM = AN$, and $CN = CP$, we infer that the three points M, N, P are the points of contact of the ⊙ inscribed in the $\triangle ABC$ (§ 246), whence the construction is obvious.



Construction. Inscribe a ⊙ in the $\triangle ABC$ (§ 288), and mark the points of contact M, N, P. With centre A and radius AM describe a ⊙; with centre B and radius BM describe a ⊙; with centre C and radius CN describe a ⊙. These Ⓢ are the Ⓢ required.

Ex. 210. To draw a common tangent to two given circles.

ANALYSIS. Suppose the problem solved, and let O, O' be the centres of the ⊙s, AB one of the common tangents; also let r and r' denote the radii of the ⊙s. Join OA, $O'B$; then OA and $O'B$ are ⊥ to AB (§ 240), therefore ∥ to each other (§ 100). From A lay off upon AO a length $AM = O'B$, and join MO'; then the figure $ABO'M$ is a ▱ (§ 182) and a rectangle. And since $O'M$ is ⊥ to OA, and $OM = r - r'$, therefore $O'M$ will touch a ⊙ having O for centre and $r - r'$ for radius. This ⊙ can be described and $O'M$ drawn; after which the remainder of the construction is obvious.



CONSTRUCTION. With centre O and radius $r - r'$ describe a ⊙. Draw to this ⊙ from O' the tangents $O'M$, $O'N'$ (§ 287). Join OM, ON, and produce them to meet the given circumference at A and C. Draw the radii $O'B$ ∥ to OA, and $O'D$ ∥ to OC. Draw straight lines through A and B, and through C and D. These lines are tangents to the given ⊙s.

PROOF.	MA is = and ∥ to $O'B$.	(Cons.)
	∴ AB is ∥ to MO'.	(§ 182)
	$\angle OMO' = 90°$.	(§ 240)
	∴ $\angle OAB = 90°$.	(§ 102)
And	$O'B$ is ∥ to OA.	
	∴ $\angle O'BA = 90°$.	
	∴ AB touches each ⊙.	(§ 239)

Similarly, CD may be shown to be a tangent.

DISCUSSION. Let d denote the distance OO' between the centres. If (as in the figure) $d > r + r'$, two more common tangents EF, GH may be drawn by the use of an auxiliary ⊙ having for radius $r + r'$ instead of $r - r'$. The figure shows the details of their construction. AB and CD are called *exterior* tangents, EF and GH *interior* tangents. The exterior tangents meet at a point P which lies in the line of centres OO'

produced. The interior tangents meet at a point Q which lies in the line of centres between O and O'. P is called the *direct centre of similitude*, Q the *inverse centre of similitude.*

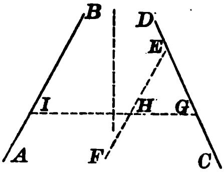
If $d = r + r'$, the ⊛ touch externally, and *three* common tangents can be drawn.

If $d < r + r'$ but $> r - r'$, the ⊛ intersect, and *two* common tangents can be drawn.

If $d = r - r'$, the ⊛ touch internally, and *one* common tangent can be drawn.

If $d < r - r'$, one ⊙ lies within the other without touching it, and the solution of the problem is impossible.

Ex. 211. To bisect the angle formed by two lines, without producing the lines to their point of intersection.



CONSTRUCTION. Let AB, CD be the lines. Through any point E of CD draw $EF \parallel$ to BA (§ 278).

Upon the sides of the $\angle FEC$ take equal lengths EH, EG, and produce GH to meet AB at I.

Draw the $\perp$ bisector of GI; it will bisect the $\angle$ formed by the lines AB, CD.

PROOF. The $\angle EGH = \angle EHG$. (§ 154)

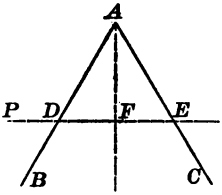
Also $\angle EHG = \angle BIH$. (§ 106)

$\therefore \angle EGH = \angle BIG$.

∴ the △ formed by the two given lines and the line GI is isosceles (§ 156).

∴ the $\perp$ bisector of GI bisects the $\angle$ of AB and CD (Ex. 15).

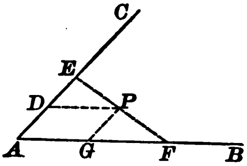
Ex. 212. To draw a line through a given point, so that it shall form with the sides of a given angle an isosceles triangle.



ANALYSIS. Let $\angle BAC$ be the given $\angle$, P the given point, and suppose the problem solved, and PDE the line required. If we draw AF bisecting $\angle DAE$, then AF must be $\perp$ to PE (Ex. 14).

CONSTRUCTION. Draw the bisector AF of the $\angle BAC$ (§ 275). Draw $PE \perp$ to AF (§ 272), and let it cut AB in D, AC in E. PDE is the line required.

Ex. 213. Given a point P between the sides of an angle BAC. To draw through P a line terminated by the sides of the angle and bisected at P.

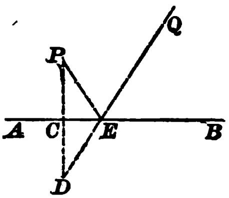


Analysis. Suppose the problem solved, and EF the required line. If we draw PD $\parallel$ to BA, and meeting AC at D, then $AD = DE$ (§ 188). Hence the point E may be determined.

Construction. Draw PD as described. Upon AC take $DE = AD$. Draw EP, and produce it to meet AB at F. EF is the line required.

Ex. 214. Given two points P, Q, and a line AB; to draw lines from P and Q which shall meet on AB and make equal angles with AB.

Analysis. Let PE, QE be the lines required. Draw $PC \perp$ to AB, and let QE meet PC produced at D.



Since $\angle QEB = \angle DEC$,

we have $\angle DEC = \angle PEC$.

$\therefore \triangle PCE = \triangle DCE$, (§ 149)

and $PC = CD$.

Hence the following construction:

Construction. Draw $PC \perp$ to AB, and produce it, making $CD = PC$. Join QD, cutting AB at E. Join PE. PE and QE are the required lines.

Proof. In the rt. $\triangle$ PCE, DCE,

CE is common, and $PC = CD$. (Cons.)

$\therefore \triangle PCE = \triangle DCE$, (§ 151)

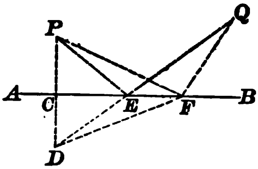
and $\angle PEC = \angle DEC$.

But $\angle DEC = \angle QEB$. $\therefore \angle PEC = \angle QEB$.

Discussion. If the line AB passes between P and Q, the solution is evidently the line joining P and Q.

Ex. 215. To find the shortest path from P to Q which shall touch a line AB.

Analysis. Let E be a point in AB, such that PE and QE make equal $\angle$ with AB (see Ex. 214), and let F be any other point in AB. Draw $PC \perp$ to AB, produce it to D, making $CD = PC$. Join PF, QF, DE, DF.



Then $\triangle PCE = \triangle DCE$, and $\triangle PCF = \triangle DCF$. (§ 151)

$\therefore PE = DE$, and $PF = DF$.

$\therefore PE + EQ = DE + EQ = DQ$,

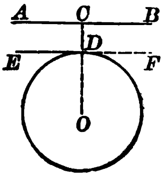
and $PF + FQ = DF + FQ$.

But $DQ < DF + FQ$. (§ 33)

$\therefore PE + EQ < PF + FQ$.

Hence the shortest path from P to Q which can be drawn so as to touch the line AB is the broken line whose two parts form equal $\angle s$ with AB.

Ex. 216. To draw a tangent to a given circle, so that it shall be parallel to a given straight line.



Construction. Let O be the centre of the given ⊙, AB the given line. Draw $OC \perp AB$, meeting the ⊙ at D. Draw $EDF \perp$ to OC.

Then EF is the tangent required.

Proof. EF is a tangent to the ⊙. (§ 239)

EF is $\parallel$ to AB. (§ 100)

Put the detail

BOOK III.

Ex. 217. Construct x, if (1) $x = \frac{ab}{c}$, (2) $x = \frac{a^2}{c}$.

Special Cases: (1) $a = 2$, $b = 3$, $c = 4$; (2) $a = 3$, $b = 7$, $c = 11$; (3) $a = 2$, $c = 3$; (4) $a = 3$, $c = 5$; (5) $a = 2c$.

(1) $$x = \frac{ab}{c}.$$

SOLUTION. $$cx = ab.$$

$$\therefore c : a = b : x. \qquad (§ 297)$$

Find x, a fourth proportional to c, a, and b. (§ 352)

(2) $$x = \frac{a^2}{c}.$$

$$cx = a^2.$$

$$\therefore c : a = a : x. \qquad (§ 297)$$

Find x, a third proportional to c and a. (§ 353)

Special Cases: (1) $a = 2$, $b = 3$, $c = 4$.

Using a convenient unit, solve like (1).

$$x = 1\tfrac{1}{2}.$$

(2) Like (1). $x = 1\tfrac{10}{11}$.

(3) $a = 2$, $c = 3$.

Find x, a third proportional to c and a. (§ 353)

$$x = \tfrac{4}{3}.$$

(4) Like (3). $x = 1\tfrac{4}{5}$.

(5) Like (3). $x = 4c$.

Ex. 218. Construct x if $x = \sqrt{ab}$.

Special Cases: (1) $a = 2$, $b = 3$; (2) $a = 1$, $b = 5$; (3) $a = 3$, $b = 7$.

$$x = \sqrt{ab}.$$

SOLUTION. Find x, a mean proportional between a and b. (§ 354)

Special Cases: (1) $a = 2$, $b = 3$.

Using a convenient unit, find x, a mean proportional between 2 and 3. (§ 354.)

$$x = \sqrt{6}.$$

(2) Like (1). $x = \sqrt{5}.$

(3) Like (1). $x = \sqrt{21}.$

Ex. 223. If the sides of a triangle are 3, 4, and 5, what kind of an angle is opposite 5?

A right angle. For $3^2 + 4^2 = 5^2$. (§ 338)

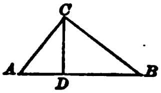
Ex. 224. If the sides of a triangle are 7, 9, and 12, what kind of an angle is opposite 12?

An obtuse angle. For $12^2 > 7^2 + 9^2$. (§ 343)

Ex. 225. If the sides of a triangle are 7, 9, and 11, what kind of an angle is opposite 11?

An acute angle. $11^2 < 7^2 + 9^2$. (§ 342)

Ex. 226. The legs of a right triangle are 8 inches and 12 inches; find the lengths of the projections of these legs upon the hypotenuse, and the distance of the vertex of the right angle from the hypotenuse.



In △ ABC, $AC = 8$ and $BC = 12$.

To find AD, DB, and DC.

SOLUTION. $AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{8^2 + 12^2} = 14.422.$ (§ 338)

$$AB : AC = AC : AD, \quad (§ 334, \text{II.})$$

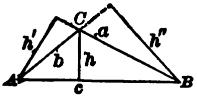
$$14.422 : 8 = 8 : AD,$$

$$AD = 4.438.$$

$$DB = AB - AD = 9.984.$$

$$DC = \sqrt{AD \times DB} = \sqrt{4.438 \times 9.984} = 6.656. \quad (§ 334, \text{I.})$$

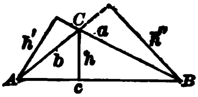
Ex. 227. If the sides of a triangle are 6 inches, 9 inches, and 12 inches, find the lengths (1) of the altitudes; (2) of the medians; (3) of the bisectors; (4) of the radius of the circumscribed circle.



In △ ABC, $a = 9$, $b = 6$, $c = 12$.

(1) Let h, h', and h'' be altitudes drawn to c, a, and b, respectively.

To find the lengths of h, h', and h''.



Solution. $h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$

$= \frac{2}{12}\sqrt{\frac{27}{2}(\frac{27}{2}-9)(\frac{27}{2}-6)(\frac{27}{2}-12)}$ (Ex. 219)

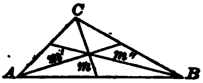
$= 4.357.$

$h' = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2}{9}\sqrt{\frac{27}{2}(\frac{27}{2}-9)(\frac{27}{2}-6)(\frac{27}{2}-12)} = 5.809.$

$h'' = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2}{6}\sqrt{\frac{27}{2}(\frac{27}{2}-9)(\frac{27}{2}-6)(\frac{27}{2}-12)} = 8.714.$

(2) Let m, m', and m'' be medians drawn to c, a, and b, respectively.

To find the lengths of m, m', and m''.



Solution. $m = \frac{1}{2}\sqrt{2(a^2+b^2)-c^2} = \frac{1}{2}\sqrt{2(9^2+6^2)-12^2}$

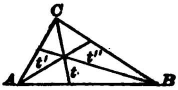
$= 4.743.$ (Ex. 220)

$m' = \frac{1}{2}\sqrt{2(b^2+c^2)-a^2} = \frac{1}{2}\sqrt{2(6^2+12^2)-9^2} = 8.351.$

$m'' = \frac{1}{2}\sqrt{2(a^2+c^2)-b^2} = \frac{1}{2}\sqrt{2(9^2+12^2)-6^2} = 10.173.$

(3) Let t, t', and t'' be the bisectors of ∠s C, A, and B, respectively.

To find the lengths of t, t', t''.



Solution. $t = \frac{2}{a+b}\sqrt{abs(s-c)} = \frac{2}{9+6}\sqrt{9\times 6\times \frac{27}{2}(\frac{27}{2}-12)}$

$= 4.409.$ (Ex. 221)

$t' = \frac{2}{b+c}\sqrt{bcs(s-a)} = \frac{2}{6+12}\sqrt{6\times 12\times \frac{27}{2}(\frac{27}{2}-9)}$

$= 7.348.$

$t'' = \frac{2}{a+c}\sqrt{acs(s-b)} = \frac{2}{9+12}\sqrt{9\times 12\times \frac{27}{2}(\frac{27}{2}-6)}$

$= 9.959.$

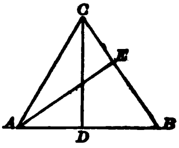
(4) Let R be the radius of the circumscribed ⊙.

To find the length of R.

Solution. $R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$ (Ex. 222)

$= \frac{9\times 6\times 12}{4\sqrt{\frac{27}{2}(\frac{27}{2}-9)(\frac{27}{2}-6)(\frac{27}{2}-12)}} = 6.196.$

Ex. 228. Any two altitudes of a triangle are inversely proportional to the corresponding bases.



In the △ ABC let AE and CD be the altitudes drawn to BC and AB respectively.

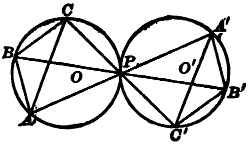
To prove $AE : CD = AB : BC$.

Proof. The rt. ▲ AEB and BDC have the common acute ∠ B.

$\therefore$ ▲ AEB and BDC are similar. (§ 323)

Whence $AE : CD = AB : BC$.

Ex. 229. Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A', B', C', respectively. Prove that the triangles ABC, $A'B'C'$ are similar.



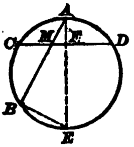
In the ⊙ O and O' let AA', BB', and CC' be drawn through P, the point of contact, and let AC, BC, AB, $A'C'$, $B'C'$, and $A'B'$ be drawn.

To prove ▲ ABC and $A'B'C'$ similar.

Proof. AC, BC, and AB are ∥ to $C'A'$, $C'B'$, and $B'A'$, respectively. (Ex. 86)

$\therefore$ ▲ ABC and $A'B'C'$ are similar. (§ 327)

Ex. 230. Two chords AB, CD intersect at M, and A is the middle point of the arc CD. Prove that the product $AB \times AM$ remains the same if the chord AB is made to turn about the fixed point A.



Let AB and CD be two chords intersecting at M, and A be the middle point of the arc CD.

To prove that $AB \times AM$ is constant.

Proof. Draw the diameter AE and join BE.

AF is ⊥ to CD. (§ 232)

△ ABE is a rt. △. (§ 264)

The rt. ▲ AFM and ABE have the common ∠ MAF.

$\therefore$ ▲ AFM and ABE are similar. (§ 323)

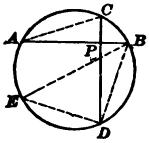
Whence $AF : AB = AM : AE$.

$\therefore AF \times AE = AB \times AM$. (§ 295)

But $AF \times AE$ is constant.

$\therefore AB \times AM$ is constant.

Ex. 231. The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.



Let the ⊥ chords *AB* and *CD* intersect at *P*. Draw the diameter *BE*.

Then $\overline{PA}^2 + \overline{PC}^2 + \overline{PB}^2 + \overline{PD}^2 = \overline{BE}^2.$

Proof. Join *AC*, *ED*, and *BD*.

arc *AC* + arc *BD* = arc *BDE*. (Ex. 84)

Subtract arc *BD* from both sides.

Then arc *AC* = arc *ED*.

$\therefore AC = ED.$ (§ 230)

Now, $\overline{ED}^2 + \overline{BD}^2 = \overline{BE}^2.$ (§ 338)

Substituting *AC* for *ED*,

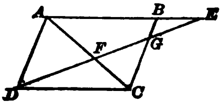
$\overline{AC}^2 + \overline{BD}^2 = \overline{BE}^2.$

But $\overline{AC}^2 = \overline{PA}^2 + \overline{PC}^2,$

and $\overline{BD}^2 = \overline{PB}^2 + \overline{PD}^2.$ (§ 338)

Substituting, $\overline{PA}^2 + \overline{PC}^2 + \overline{PB}^2 + \overline{PD}^2 = \overline{BE}^2.$

Ex. 232. In a parallelogram *ABCD*, a line *DE* is drawn, meeting the diagonal *AC* in *F*, the side *BC* in *G*, and the side *AB* produced in *E*. Prove that $\overline{DF}^2 = FG \times FE.$



Let *ABCD* be a ▱, and *DE* be drawn, meeting the diagonal *AC* in *F*, the side *BC* in *G*, and the side *AB* produced in *E*.

Then $\overline{DF}^2 = FG \times FE.$

Proof. In the ▲ *AEF* and *DFC*,

$\angle AEF = \angle FDC,$

and $\angle FAE = \angle FCD.$ (§ 104)

∴ ▲ *AEF* and *DFC* are similar. (§ 322)

Whence $DF : FE = FC : AF.$

In the ▲ *AFD* and *GFC*,

$\angle AFD = \angle GFC,$ (§ 95)

and $\angle DAF = \angle FCG.$ (§ 104)

∴ ▲ *AFD* and *GFC* are similar. (§ 322)

Whence $FG : DF = FC : AF.$

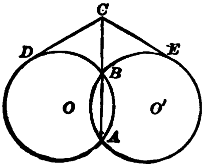
But $DF : FE = FC : AF.$

$\therefore DF : FE = FG : DF.$

$\therefore \overline{DF}^2 = FG \times FE.$ (§ 295)

If 2 ⊙ are tangent to each other externally at A, common tangent thro A bisects other 2 common tangents.

Ex. 233. The tangents to two intersecting circles drawn from any point in their common chord produced are equal.



Let O and O' be two intersecting ⊙s, ABC be their common chord produced, and CD and CE be tangents to O and O'.

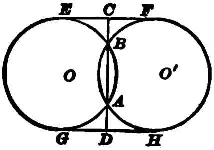
Then $CD = CE$.

PROOF. $\overline{CD}^2 = CA \times CB$,

and $\overline{CE}^2 = CA \times CB$. (§ 348)

$\therefore \overline{CD}^2 = \overline{CE}^2$ and $CD = CE$. (Ax. 1)

Ex. 234. The common chord of two intersecting circles, if produced, will bisect their common tangents.



Let O and O' be two intersecting ⊙s, DC their common chord produced, and EF and GH their common tangents.

Then DC bisects EF and GH.

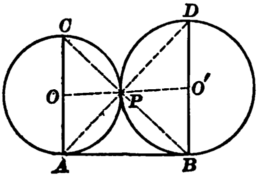
PROOF. $\overline{EC}^2 = CB \times CA$,

and $\overline{FC}^2 = CB \times CA$. (§ 348)

$\therefore \overline{EC}^2 = \overline{FC}^2$ and $EC = FC$. (Ax. 1)

In the same way, it may be proved that GH is bisected at D.

Ex. 235. If two circles touch each other, their common tangent is a mean proportional between their diameters.



Let O and O' be two ⊙s touching at P, and AB their common tangent.

Then $AC : AB = AB : BD$.

PROOF. Draw AP, CP, BP, DP, OO'.

Since AC and BD are ⊥ to AB (§ 240), they are ∥ (§ 100).

$\therefore \angle AOP + \angle BO'P = 2$ rt. ∠s. (§ 109)

But $\angle PAB = \frac{1}{2} \angle AOP$, and $\angle PBA = \frac{1}{2} \angle BO'P$. (§§ 262, 269)

$\therefore \angle PAB + \angle PBA =$ a rt. ∠; $\therefore \angle APB =$ a rt. ∠. (§ 138)

Also $\angle APC =$ a rt. ∠ (§ 264); $\therefore CPB$ is a straight line. (§ 93)

In like manner DPA is a straight line.

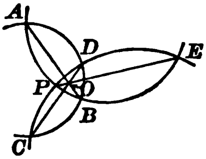
Now in the rt. △s ACB and ADB,

$\angle C = \angle DAB$. (§§ 263, 269)

$\therefore$ △s ACB and ADB are similar. (§ 323)

Whence $AC : AB = AB : BD$.

Ex. 236. If three circles intersect one another, the common chords all pass through the same point.



Let the $\circledS$ ABC, ABE, and CDE intersect one another, and AB and CD meet at O. Join EO. Suppose that EO produced meets ABE at P and CDE at Q.

Then P and Q are the same point.

Proof. In $\odot ABC$, $OA \times OB = OC \times OD$. (§ 345)

In $\odot ABE$, $OA \times OB = OE \times OP$.

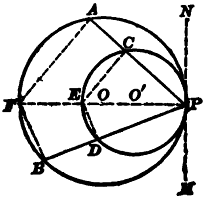
In $\odot CDE$, $OC \times OD = OE \times OQ$.

Whence $OE \times OP = OE \times OQ$.

$\therefore OP = OQ$.

Therefore EP is the common chord of $\circledS$ ABE and CDE, and passes through the point O.

Ex. 237. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller circle.



Let the $\circledS$ O and O' be tangent internally at P, and chords PA and PB be drawn, cutting the circumference of O' at C and D.

Then $PC : PA = PD : PB$.

Proof. Draw the tangent MN at the point P.

Draw $PF \perp$ to MN.

Then PF passes through the centres O and O'. (§ 241)

Join EC, ED, FA, and FB.

$\angle$s PCE and PAF are rt. $\angle$s. (§ 264)

The rt. $\triangle$s PCE and PAF have the $\angle EPC$ common.

$\therefore$ $\triangle$s PCE and PAF are similar. (§ 323)

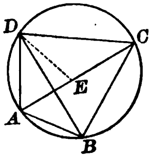
Whence $PC : PA = PE : PF$.

In the same way, it may be proved that

$PD : PB = PE : PF$.

$\therefore PC : PA = PD : PB$.

Ex. 238. In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.



Let $ABCD$ be an inscribed quadrilateral, and AC and DB its diagonals.

Then $\quad AC \times DB = DC \times AB + AD \times BC.$

Proof. Draw DE, making $\angle CDE = \angle ADB$.

Add to each $\angle BDE$; then in $\triangle$ ADE and BDC,

$$\angle ADE = \angle BDC, \qquad \text{(Ax. 2)}$$

and $$\angle DAC = \angle DBC. \qquad \text{(§ 267)}$$

$\therefore$ $\triangle$ ADE and BDC are similar. (§ 322)

Whence $\quad AD : DB = AE : BC.$

$$\therefore AD \times BC = DB \times AE. \qquad (1)$$

In $\triangle$ DEC and ABD, $\angle CDE = \angle ADB$, (Cons.)

and $$\angle DCE = \angle ABD. \qquad \text{(§ 267)}$$

$\therefore$ $\triangle$ DEC and ABD are similar. (§ 322)

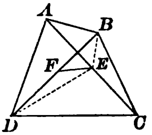
Whence $\quad EC : AB = DC : DB.$

$$\therefore \quad DC \times AB = DB \times EC. \qquad (2)$$

Add (1) and (2); then

$$DC \times AB + AD \times BC = DB(AE + EC) = DB \times AC.$$

Ex. 239. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the line joining the middle points of the diagonals.



Let $ABCD$ be a quadrilateral, and EF the line joining the middle points of the diagonals AC and BD.

Then $\quad \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2.$

Proof. Draw BE and DE.

$$\overline{AB}^2 + \overline{BC}^2 = 2\left(\frac{AC}{2}\right)^2 + 2\,\overline{BE}^2, \qquad \text{(§ 344)}$$

and $$\overline{CD}^2 + \overline{DA}^2 = 2\left(\frac{AC}{2}\right)^2 + 2\,\overline{DE}^2. \qquad \text{(§ 344)}$$

Adding these equalities,

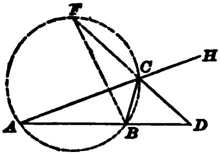
$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\left(\frac{AC}{2}\right)^2 + 2(\overline{BE}^2 + \overline{DE}^2).$$

But $$\overline{BE}^2 + \overline{DE}^2 = 2\left(\frac{BD}{2}\right)^2 + 2\,\overline{EF}^2. \qquad \text{(§ 344)}$$

$$\therefore \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\left(\frac{AC}{2}\right)^2 + 4\left(\frac{BD}{2}\right)^2 + 4\,\overline{EF}^2$$

$$= \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2.$$

Ex. 240. The square of the bisector of an exterior angle of a triangle is equal to the product of the external segments determined by the bisector upon one of the sides diminished by the product of the other two sides.



Let ABC be a $\triangle$, and CD be the bisector of the exterior $\angle BCH$, meeting AB produced at D.

Then $\overline{CD}^2 = AD \times BD - AC \times BC$.

Proof. Circumscribe a $\odot$ about the $\triangle ABC$, produce DC to meet the circumference in F, and draw BF.

In $\triangle$s ACD and BCF,

$$\angle BAC = \angle BFC, \qquad (\S\ 267)$$

$$\angle FCA = \angle HCD. \qquad (\S\ 95)$$

But $$\angle HCD = \angle BCD. \qquad \text{(Hyp.)}$$

$$\therefore \angle FCA = \angle BCD. \qquad \text{(Ax. 1)}$$

Add to each $\angle ACB$; then

$$\angle FCB = \angle ACD. \qquad \text{(Ax. 2)}$$

$$\therefore \triangle\text{s } ACD \text{ and } BCF \text{ are similar.} \qquad (\S\ 322)$$

Whence $$CD : BC = AC : FC.$$

$$\therefore FC \times CD = AC \times BC.$$

Now $$AD \times BD = FD \times CD \qquad (\S\ 347)$$

$$= (FC + CD)\, CD$$

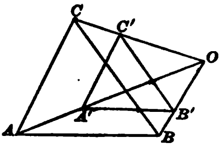
$$= FC \times CD + \overline{CD}^2.$$

But $$FC \times CD = AC \times BC.$$

$$\therefore AD \times BD = AC \times BC + \overline{CD}^2.$$

Transposing, $$\overline{CD}^2 = AD \times BD - AC \times BC.$$

Ex. 241. If a point O is joined to the vertices of a triangle ABC, and through any point A' in OA a line parallel to AB is drawn, meeting OB at B', and then through B' a line parallel to BC, meeting OC at C', and C' is joined to A', the triangle $A'B'C'$ will be similar to the triangle ABC.



Let ABC be a $\triangle$, O any point; let OA, OB, and OC be drawn; from A', any point in OA, let $A'B'$ be drawn $\parallel$ to AB; let $B'C'$ be drawn $\parallel$ to BC; join C' and A'.

Then $\triangle$s ABC and $A'B'C'$ are similar.

Proof. In $\triangle$ ABC and $A'B'C'$,

$$\angle ABC = \angle A'B'C'. \qquad (§ 112)$$

In $\triangle$ AOB and $A'OB'$,

$$\angle OB'A' = \angle OBA, \text{ and } \angle OA'B' = \angle OAB. \qquad (§ 106)$$

$$\therefore \triangle\ AOB \text{ and } A'OB' \text{ are similar.} \qquad (§ 322)$$

Whence $OB' : OB = A'B' : AB.$

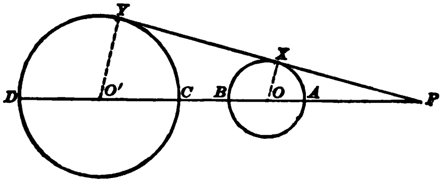
In the same way, $\triangle$ $OB'C'$ and OBC may be proved similar.

Whence $OB' : OB = B'C' : BC.$

$$\therefore A'B' : AB = B'C' : BC.$$

$$\therefore \triangle\ ABC \text{ and } A'B'C' \text{ are similar.} \qquad (§ 326)$$

Ex. 242. If the line of centres of two circles meets the circumferences at the points A, B, C, D, and meets the common exterior tangent at P, then $PA \times PD = PB \times PC$.



In the ⊙ O and O' let the line of centres meet the common tangent PXY at P and cut the circumferences at A, B, C, D.

Then $PA \times PD = PB \times PC.$

Proof. Draw the radii OX and $O'Y$.

$$\angle\ PXO \text{ and } PYO' \text{ are rt. } \angle. \qquad (§ 240)$$

The rt. $\triangle$ PXO and PYO' have the $\angle XPO$ common.

$$\therefore \triangle\ PXO \text{ and } PYO' \text{ are similar.} \qquad (§ 323)$$

Whence $PO : OX = PO' : O'Y.$

By composition and division,

$$PO + OX : PO - OX = PO' + O'Y : PO' - O'Y.$$

But $PO + OX = PB$, $PO - OX = PA$, $PO' + O'Y = PD$,

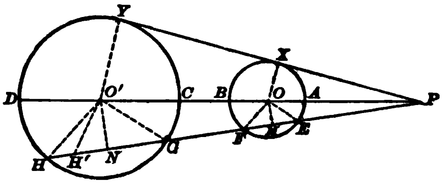
and $PO' - O'Y = PC.$

Substituting these values,

$$PB : PA = PD : PC.$$

$$\therefore PA \times PD = PB \times PC.$$

Ex. 243. The line of centres of two circles meets the common exterior tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points E, F, G, H. Prove that $PE \times PH = PF \times PG$.



In the ⊛ O and O' let the line of centres meet the common tangent PXY at P, and cut the circumferences at A, B, C, D; let the secant PH be drawn, cutting the circumferences at E, F, G, H.

Then $$PE \times PH = PF \times PG.$$

Proof. Draw the radii OX, OE, OF, $O'Y$, $O'G$, and $O'H$; draw OM and $O'N \perp$ to PH.

rt. △ POM and $PO'N$ have the common $\angle OPM$ and are similar. (§ 323)

Whence $$PO : PO' = OM : O'N.$$

But $$PO : PO' = OX : O'Y = OF : O'H. \qquad \text{(Ex. 242)}$$

$$\therefore OM : O'N = OF : O'H.$$

Suppose $O'H'$ be drawn $\parallel$ to OF.

Then △ OMF and $O'NH'$ are similar. (§ 327)

Whence $$OM : O'N = OF : O'H'.$$

$$\therefore O'H = O'H' \text{ and coincides with it.}$$

$$\therefore O'H \text{ is } \parallel \text{ to } OF.$$

Whence $$PO : PO' = PF : PH. \qquad \text{(§ 309)}$$

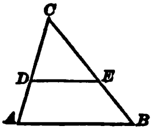
In the same way, it may be proved that

$$PO : PO' = PE : PG.$$

$$\therefore PE : PG = PF : PH.$$

$$\therefore PE \times PH = PF \times PG.$$

Ex. 244. A line is drawn parallel to a side AB of a triangle ABC, and cutting AC in D, BC in E. If $AD : DC = 2 : 3$, and $AB = 20$ inches, find DE.



SOLUTION. $AD : DC = 2 : 3.$

By composition, $AD + DC : DC = 2 + 3 : 3.$

Or $AC : DC = 5 : 3.$

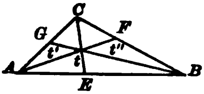
But $AC : DC = AB : DE.$

$\therefore AB : DE = 5 : 3$

Or $20 : DE = 5 : 3.$

$\therefore DE = 12$ inches.

Ex. 245. The sides of a triangle are 9, 12, 15. Find the segments made by bisecting the angles.



In the $\triangle ABC$ let t, t', and t'' be the bisectors of $\angle$s C, A, and B, respectively.

$$AB = 15,\ BC = 12,\ AC = 9.$$

To find the lengths of AE, EB, BF, FC, CG, and GA.

SOLUTION. $AE : EB = AC : BC = 9 : 12.$ (§ 313)

$$\therefore AE = \tfrac{9}{21} \times 15 = 6\tfrac{3}{7},$$

$$EB = \tfrac{12}{21} \times 15 = 8\tfrac{4}{7}.$$

In the same way it may be proved that

$$BF = \tfrac{15}{24} \times 12 = 7\tfrac{1}{2},$$

$$FC = \tfrac{9}{24} \times 12 = 4\tfrac{1}{2},$$

$$CG = \tfrac{12}{27} \times 9 = 4,$$

$$GA = \tfrac{15}{27} \times 9 = 5.$$

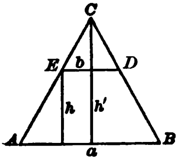
Ex. 246. A tree casts a shadow 90 feet long when a vertical rod 6 feet high casts a shadow 4 feet long. How high is the tree?

SOLUTION. Let $x =$ the height of the tree.

Then $4 : 90 = 6 : x.$

$\therefore x = 135$ feet.

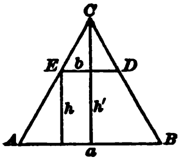
Ex. 247. The bases of a trapezoid are represented by a, b, and the altitude by h. Find the altitudes of the two triangles formed by producing the legs till they meet.



Let ABC be a $\triangle$ formed by producing the legs of the trapezoid $ABDE$, h and h' be the altitudes of $ABDE$ and ABC, respectively, and a and b be the bases of the trapezoid.

To find the altitudes of $\triangle$s ABC and EDC in terms of h, a, and b.

SOLUTION. The altitude of $\triangle EDC = h' - h$.



Since AB and ED are $\parallel$,

$$h' - h : h' = b : a. \qquad (§\ 328)$$

$$\therefore ah' - ah = bh'.$$

$$\therefore h' = \frac{ah}{a-b},$$

and

$$h' - h = \frac{ah}{a-b} - h = \frac{bh}{a-b}.$$

Ex. 248. The sides of a triangle are 6, 7, 8. In a similar triangle the side homologous to 8 is equal to 40. Find the other two sides.

SOLUTION. Let $x =$ the side homologous to 6,

$y =$ the side homologous to 7.

Then $6 : 8 = x : 40,$ (§ 319)

also $7 : 8 = y : 40.$

$\therefore x = 30$, and $y = 35$.

Ex. 249. The perimeters of two similar polygons are 200 feet and 300 feet. If a side of the first polygon is 24 feet, find the homologous side of the second polygon.

SOLUTION. Let $x =$ a side of the second polygon homologous to 24.

Then $200 : 300 = 24 : x.$ (§ 333)

$\therefore x = 36$ feet.

Ex. 250. How long must a ladder be to reach a window 24 feet high, if the lower end of the ladder is 10 feet from the side of the house?

SOLUTION. Let $x =$ the length of the ladder.

Then $x = \sqrt{24^2 + 10^2} = 26$ feet. (§ 338)

Ex. 251. If the side of an equilateral triangle $= 8$, find the altitude.

SOLUTION. Let $x =$ the length of the altitude.

Then $x = \sqrt{a^2 - (\frac{1}{2}a)^2} = \frac{a}{2}\sqrt{3}.$ (§ 338)

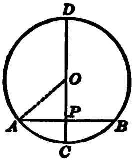
Ex. 252. If the altitude of the equilateral triangle $= h$, find the side.

Let a be the side of the equilateral $\triangle$.

SOLUTION. $a^2 - (\frac{1}{2}a)^2 = h^2.$ (§ 338)

$$\therefore a^2 = \frac{4h^2}{3}, \text{ and } a = \frac{2h}{3}\sqrt{3}.$$

Ex. 253. Find the lengths of the longest and the shortest chord that can be drawn through a point 6 inches from the centre of a circle whose radius is equal to 10 inches.



Let P be the given point, CD a diameter drawn through P, and AB a chord ⊥ to CD.

Solution. Draw AO.

CD is the longest chord and = 20 inches. (§ 227)

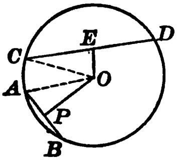
$$AP = \sqrt{\overline{AO}^2 - \overline{OP}^2} = \sqrt{10^2 - 6^2} = 8. \quad (§\ 339)$$

But $AP = \frac{1}{2} AB$. (§ 232)

$\therefore AB = 16$ inches.

AB is the shortest chord. (Ex. 115)

Ex. 254. The distance from the centre of a circle to a chord 10 inches long is 12 inches. Find the distance from the centre to a chord 24 inches long.



Let AB be 10 inches, CD 24 inches, OP ⊥ to AB 12 inches.

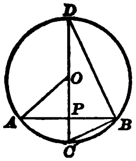
To find the length of OE ⊥ to CD.

Solution. Draw OA and OC.

$$OA = \sqrt{\overline{OP}^2 + \overline{AP}^2} = \sqrt{12^2 + 5^2} = 13 = OC, \quad (§\ 338)$$

$$OE = \sqrt{\overline{OC}^2 - \overline{CE}^2} = \sqrt{13^2 - 12^2} = 5 \text{ inches.}$$

Ex. 255. The radius of a circle is 5 inches. Through a point 3 inches from the centre a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance from one end of the chord to the ends of the diameter.



Let AB be a chord drawn ⊥ to the diameter CD and 3 inches from the centre O.

To find the lengths of AB, BD, and BC.

Solution. Draw OA.

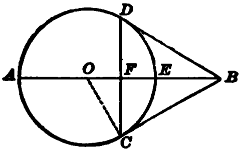
$$AP = \sqrt{\overline{OA}^2 - \overline{OP}^2} = \sqrt{5^2 - 3^2} = 4.$$

$$\therefore AB = 2 \times AP = 8 \text{ inches.}$$

$$BD = \sqrt{\overline{PB}^2 + \overline{PD}^2} = \sqrt{4^2 + 8^2} = 8.944 \text{ inches.}$$

$$BC = \sqrt{\overline{PB}^2 + \overline{PC}^2} = \sqrt{4^2 + 2^2} = 4.472 \text{ inches.}$$

Ex. 256. The radius of a circle is 6 inches. Through a point 10 inches from the centre tangents are drawn. Find the lengths of the tangents, and also of the chord joining the points of contact.



Let AB be a secant drawn through the centre O, BD and BC be tangents, and DC be a chord joining D and C.

To find the lengths of BC, BD, and DC.

SOLUTION. $AB = 16,\ EB = 4.$

$$\overline{BC}^2 = AB \times EB \ (\S\ 348) = 16 \times 4.$$

$$\therefore BC = 8 \text{ inches} = BD. \qquad (\S\ 246)$$

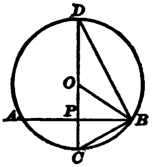
$$OB = 10, \quad OF : FB = \overline{OC}^2 : \overline{BC}^2 = 36 : 64. \qquad (\S\ 335)$$

$$\therefore FB = \tfrac{64}{100} \times 10 = 6.4.$$

$$CF = \sqrt{\overline{BC}^2 - \overline{FB}^2} = \sqrt{8^2 - 6.4^2} \ (\S\ 338) = 4.8.$$

$$\therefore DC = 2 \times CF = 9.6 \text{ inches}. \qquad (\S\ 232)$$

Ex. 257. If a chord 8 inches long is 3 inches distant from the centre of the circle, find the radius and the distances from the end of the chord to the ends of the diameter which bisects the chord.



Let AB be a chord 8 inches long, 3 inches distant from O, and CD be a diameter bisecting AB.

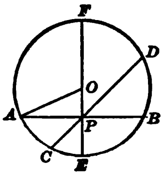
To find the lengths of OB, BD, and BC.

SOLUTION. $OB = \sqrt{\overline{OP}^2 + \overline{PB}^2} = \sqrt{3^2 + 4^2} = 5$ inches. (§ 338)

$$BD = \sqrt{\overline{DP}^2 + \overline{PB}^2} = \sqrt{8^2 + 4^2} = 8.944 \text{ inches}.$$

$$BC = \sqrt{\overline{PB}^2 + \overline{PC}^2} = \sqrt{4^2 + 2^2} = 4.472 \text{ inches}.$$

Ex. 258. The radius of a circle is 13 inches. Through a point 5 inches from the centre any chord is drawn. What is the product of the two segments of the chord? What is the length of the shortest chord that can be drawn through the point?



Let FE be a diameter, AB be a chord $\perp$ to FE, and CPD be any other chord, $OP = 5$ inches, $OA = 13$ inches.

To find the value of $CP \times PD$, and the length of AB.

SOLUTION. $EP = 8$, $PF = 18$.

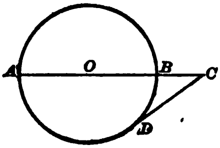
$\therefore CP \times PD = 8 \times 18 = 144.$ (§ 345)

AB is the shortest chord drawn through P. (Ex. 115)

$AP \times PB = 144$. But $AP = PB$.

$\therefore \overline{AP}^2 = 144. \quad \therefore AP = 12. \quad \therefore AB = 24$ inches.

Ex. 259. From the end of a tangent 20 inches long a secant is drawn through the centre of the circle. If the exterior segment of this secant is 8 inches, find the radius of the circle.



$CD = 20$ inches, $CB = 8$ inches.

To find the length of OB.

SOLUTION. $AC : CD = CD : CB$. (§ 348)

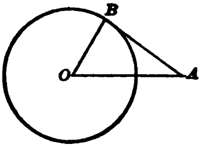
$AC : 20 = 20 : 8$.

$\therefore AC = 50$.

$AB = AC - CB = 42$.

$OB = \frac{1}{2} AB = 21$ inches.

Ex. 260. The radius of a circle is 9 inches; the length of a tangent is 12 inches. Find the length of a secant drawn from the extremity of the tangent to the centre of the circle.

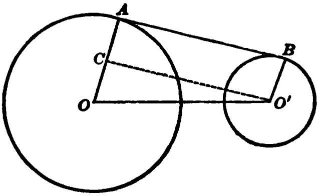


$AB = 12$ inches, $OB = 9$ inches.

To find the length of AO.

SOLUTION. $AO = \sqrt{\overline{AB}^2 + \overline{OB}^2} = \sqrt{12^2 + 9^2} = 15$ inches. (§ 338)

Ex. 261. The radii of two circles are 8 inches and 3 inches, and the distance between their centres is 15 inches. Find the lengths of their common tangents.



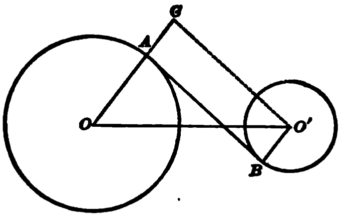
CASE I. Let AB be the exterior tangent.

To find the length of AB.

SOLUTION. Draw $O'C \parallel$ to BA.

$OO' = 15, \quad OC = 5$.

$O'C = \sqrt{\overline{OO'}^2 - \overline{OC}^2} = \sqrt{15^2 - 5^2} = 14.142$ inches $= AB$. (§ 338)



Case II. Let AB be the interior tangent.

To find the length of AB.

Solution. Make OC equal to $OA + O'B$.

Draw $O'C$.

$OC = 11, \quad OO' = 15.$

$O'C = \sqrt{\overline{OO'}^2 - \overline{OC}^2} = \sqrt{15^2 - 11^2} = 10.18$ inches $= AB$. (§ 338)

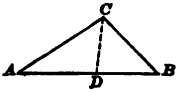
Ex. 262. Find the segments of a line 10 inches long divided in extreme and mean ratio.

Solution. $x =$ the larger segment,
$10 - x =$ the smaller segment.
Then $10 : x = x : 10 - x.$
$x^2 = 100 - 10x.$
Transposing, $x^2 + 10x = 100.$
Completing the square, $x^2 + 10x + 25 = 125.$
Extracting the square root, $x + 5 = \pm\sqrt{125}.$
$\therefore x = 6.18.$
$\therefore 10 - x = 3.82.$

Ex. 263. The sides of a triangle are 4, 5, 6. Is the largest angle acute, right, or obtuse?

An acute ∠. For $6^2 < 4^2 + 5^2$. (§ 342)

Ex. 264. To divide one side of a given triangle into segments proportional to the adjacent sides.

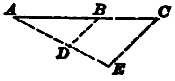


Let ABC be a △. It is required to divide AB into segments proportional to AC and BC.

Construction. Draw CD bisecting $\angle C$.

Proof. $AD : DB = AC : BC$. (§ 313)

Ex. 265. To produce a line AB to a point C so that $AB : AC = 3 : 5$.



Let AB be a straight line. It is required to produce AB to C so that $AB : AC = 3 : 5$.

Construction. Using a convenient unit, find BC, a fourth proportional to 3, AB, and 2. (§ 352)

Proof. $AD = 3, \quad AE = 5.$

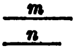
Then $AB : AC = AD : AE = 3 : 5.$ (Cons.)

the

Ex. 266. To find in one side of a given triangle a point whose distances from the other sides shall be to each other in a given ratio.

Let ABC be a $\triangle$. It is required to find a point in AB whose distances from AC and BC shall be proportional to m and : Join AD and BC.

Find BC, nt required.



$$PD : PA = PA : PC.$$

Proof

$$PA : PC = PC : PB.$$

$$\therefore PD : PA = PC : PB.$$

$$\therefore AD \text{ is } \parallel \text{ to } BC.$$

D is the point required.

$DC = 4 \cdot 2$

Proof. In the rt. $\triangle$ GHO and DFC,

$$\angle GOH = \angle DCF. \qquad (\S\ 112)$$

$$\therefore \triangle\ GHO \text{ and } DFC \text{ are similar.} \qquad (\S\ 323)$$

Whence $GH : DF = OG : CD.$

In the same way it may be proved that $\triangle$ AGO and EDC are similar.

Whence $AG : DE = OG : CD.$

$$\therefore AG : DE = GH : DF.$$

By alternation, $AG : GH = DE : DF.$

But $AG = m$, and $GH = n$.

$$\therefore DE : DF = m : n.$$

With P as a n an obtuse triangle: to draw a line from the vertex of A. Draw PA to the opposite side which shall be a mean proportional PAB is the s nents of that side.

Proof. an obtuse $\triangle$. It is required to (Cons. m C to AB which shall be a (§ 348 mean proportional between the segments of AB.

Construction. Circumscribe a ⊙ about ABC, (§ 312 and draw the radius OC; upon OC as a diameter describe a semicircle intersecting AB at P. ce $CD = PA$). Draw CP, and continue it to meet the circumference at D. Join OP.

CP is the line r $\therefore AB = P$

Proof. $\therefore PA : AB = A$. $\angle$. (§ 264)

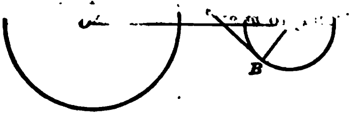
$\therefore \overline{AB}^2 = B$. (§ 232)

he problem is imposs : PB. (§ 345)

Substituting C ⊙. : PB.

Ex. 268. Through a given point P within a given circle to draw a chord AB, so that $AP : BP = 2 : 3$.

Let P be the given point. It is required to draw a chord AB through P so that $AP : BP = 2 : 3$.



CONSTRUCTIO

and n equal to w $O'C$.

long it to C, so $OO' = 15$.

OP

Find x, a fou $- \overline{OC}^2 = \sqrt{15^2 - 11^2} = 10.18$ inches $= AB$. (§ 338)

n, and the radi the segments of line 10 inches long divided

With C as a centr as a radius, cut the circumference at B. Draw CB, and $AO \parallel$ to CB. Join AP and BP.

APB is a straight line, and the chord required.

PROOF. $OP : PC = m : n = OA : CB$; (Cons.)

$\angle AOP = \angle PCB$. (§ 104)

$\therefore \triangle$ AOP and PCB are similar. (§ 326)

$\therefore \angle APO = \angle CPB$. (homologous ∠)

Since OPC is a straight line, APB is a straight line.

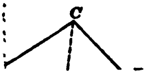
Also $AP : BP = OP : PC = m : n$.

Ex. 269. To draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.

Let P be a point in the arc AB. It is required to draw a chord through P which shall be bisected by the (§ 342)

gments propor-

tional to the adjacent sic



Let ABC be a △. I v the radius OP intersecting ial to CP. Draw $DE \parallel$ to BA

nto segments proportior hord required.

CONSTRUCTION. $PF : FE = CP : CD$. (§ 309)

PROOF. $CP = CD$. $\therefore PF = FE$.

Ex. 265. To produce

Ex. 270. To draw through a point P, exterior to a given circle, a secant PAB so that $PA : AB = ~~4 : 3~~$.

$= 3 : 5$.

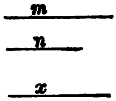
Let P be a point exterior to the $\odot$ ABC. It is required to draw through P a secant PAB, so that $PA : AB = m : n = 4 : 3$.

ient unit, find BC,

nd 2.

3, $AE = 5$.

$AD : AE = 3 : 5$.



Construction. Draw the tangent PC.

Divide PC so that $PD : DC = 4 : 3$. (§ 351)

Construction. I a mean proportional between PD and PC. (§ 354)
ngle with AB. D re with a radius equal to x, cut the circumference at
AE : Join AD and BC.
Find BC, nt required.

$$PD : PA = PA : PC. \quad \text{(Cons.)}$$

Proof

$$PA : PC = PC : PB. \quad (\S\ 348)$$

$$\therefore PD : PA = PC : PB.$$

$$\therefore AD \text{ is } \parallel \text{ to } BC. \quad (\S\ 312)$$

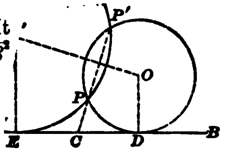
$$\therefore PA : AB = PD : DC = 4 : 3.$$

Discussion. The problem is impossible when the distance from P to the nearest point of the circumference is greater than $\frac{4}{3}$ of the diameter of the ⊙.

from an exterior

s required to construct a ⊙ gh a point P, ~~exterior to~~ a given circle, a
pass through P and P' and t $PA \times PB$.

Construction. Draw P' to the ⊙ ABD. It
AB at C. Find a mean pr PAB so that $\overline{AB}^2$
between $P'C$ and PC (§ 354)
CD equal to this mean pr
rect $OD \perp$ to AB the tangent PD.



Divide PD so that

$$PC : CD = CD : PD. \quad (\S\ 356)$$

With P as a centre with radius equal to CD, cut the circumference at A. Draw PAB, AC, and BD.

PAB is the secant required.

Proof.

$$PC : PA = PA : PD. \quad \text{(Cons.)}$$

But

$$PA : PD = PD : PB. \quad (\S\ 348)$$

$$\therefore PC : PA = PD : PB.$$

$$\therefore AC \text{ is } \parallel \text{ to } BD. \quad (\S\ 312)$$

$$\therefore PC : PA = CD : AB = PA : AB \text{ (since } CD = PA).$$

But

$$PC : PA = PA : PD.$$

$$\therefore AB = PD.$$

$$\therefore PA : AB = AB : PB.$$

$$\therefore \overline{AB}^2 = PA \times PB.$$

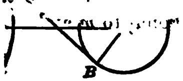
Discussion. The problem is impossible when PD is greater than the diameter of the ⊙.

Ex. 272. To find a point P in the arc subtended by a given chord AB so that $PA : PB =$ ~~3 : 1.~~

Let AB be a chord of the $\odot ABC$. It is required to find a point P in the arc AB so that

a chord AB throug

$$PA : PB = 3 : 1.$$



Construction. Through the centre O draw $EC \perp$ to AB. Take D on AB so that $AD : DB = 3 : 1$. Draw CDP, PA, and PB. P is the point required.

Proof. arc $AC =$ arc BC. (§ 232)

$\therefore \angle APC = \angle BPC$ (being measured by halves of equal arcs).

$\therefore AP : PB = AD : DB = 3 : 1.$ (§ 313)

Ex. 273. To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be to each other in the ratio of ~~3 : 5.~~ (§ 104

Let O and O' be two Ⓢ intersecting at similar. (§ 326;

draw through P a B. (homologous $\triangle$;

Since OPC i chords formed shall ratio of 3 : 5. $'B$ is a straight line.

Also $\triangle$ the line of centres $= m : n$.

Ex. 269. To draw that chord whic. $OA : AO' = 3 : 5.$ (§ 351)

Join AP. Draw $GPF \perp$ to AP. Draw OD and $O'E \perp$ to GPF. Then GPF is the secant required.

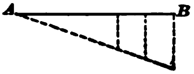
Proof. $OA : AO' = DP : PE = 3 : 5.$ (§ 311)

But $DP = \frac{1}{2}GP$ and $PE = \frac{1}{2}PF$. (§ 232)

$\therefore GP : PF = 3 : 5.$

Ex. 274. To divide a line into three parts proportional to 2, $\frac{3}{4}$, $\frac{1}{2}$.

Let AB be the given straight line. It is required to divide AB into parts proportional to 2, $\frac{3}{4}$, and $\frac{1}{2}$.



Construction. Divide AB into parts proportional to 8, 3, and 2. (§ 351)

These are the parts required.

Proof. $2 : \frac{3}{4} : \frac{1}{2} = \frac{8}{4} : \frac{3}{4} : \frac{2}{4} = 8 : 3 : 2.$

Ex. 275. Having given the greater segment of a line divided in extreme and mean ratio, to construct the line.

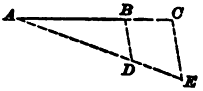
Let AB be the greater segment of a line divided in extreme and mean ratio. It is required to construct the line.

Construction. Draw AE at a convenient angle with AB. Divide AE so that

$$AE : AD = AD : DE. \qquad (\S\ 356)$$

Find BC, a fourth proportional to AD, AB, and DE.

AC is the line required.



Proof. $AC : AB = AE : AD.$ (§ 309)

But $AE : AD = AD : DE.$ (Cons.)

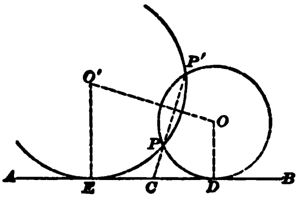
Also $AD : DE = AB : BC.$ (Cons.)

$\therefore AC : AB = AB : BC.$

Ex. 276. To construct a circle which shall pass through two given points and touch a given straight line.

Let P and P' be the two points, and AB the given straight line. It is required to construct a ⊙ which shall pass through P and P' and touch AB.

Construction. Draw $P'P$ cutting AB at C. Find a mean proportional between $P'C$ and PC (§ 354). Cut off CD equal to this mean proportional. Erect $OD \perp$ to AB, and draw $O'O$ bisecting $P'P$ at right angles. At O, the point of intersection of $O'O$ and OD, describe a circumference with a radius OP. This circumference passes through P and P', and is tangent to AB at D.



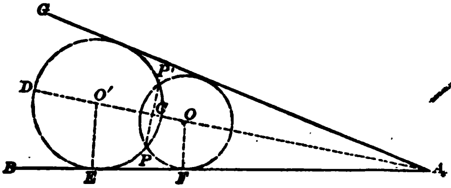
Proof. Since O is equally distant from P and P', the circumference passes through P'. Also, $P'C : CD = CD : PC.$ (Cons.)

$\therefore CD$ is tangent to the ⊙ at D. (§ 348)

Discussion. By taking CE equal to CD, a second solution may be found.

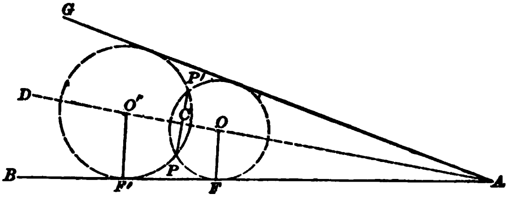
Ex. 277. To construct a circle which shall pass through a given point and touch two given straight lines.

Let AB and AG be two straight lines intersecting at A, and P be the given point.



It is required to pass a circumference through P touching AB and AG.

Construction. Draw AD bisecting $\angle BAG$. Draw $PC \perp$ to AD and prolong it to P', making CP' equal to PC. Construct a ⊙ passing through P and P' and tangent to AB (Ex. 276). This is the ⊙ required.



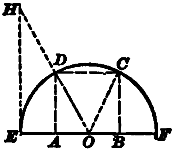
Proof. Since AD bisects PP', the centre must lie in AD. Since AD is the bisector of $\angle BAG$, the ⊙ must touch AG.

Discussion. As in Ex. 276, two solutions are possible. Special Case: when AB and AG are ∥.

Ex. 278. To inscribe a square in a semicircle.

Let $EDCF$ be a semicircle. It is required to inscribe a square in $EDCF$.

Construction. At E draw the tangent EH equal to twice the radius EO. Join HO, cutting the circumference at D. Draw $DA \perp$ to EF, $DC \parallel$ to EF, and $CB \perp$ to EF.



$ABCD$ is the square required.

Proof. $ABCD$ is a rectangle by construction. Draw OC.

In the rt. ▲ OAD and OBC,

$OD = OC$ (radii), and $AD = CB$. (§ 180)

∴ ▲ OAD and OBC are equal. (§ 161)

$\therefore OA = OB.$

In △ OEH, AD is ∥ to EH. (§ 100)

Whence $OA : AD = OE : EH.$

But $OE = \frac{1}{2} EH$ (Cons.). $\therefore OA = \frac{1}{2} AD.$

$\therefore OA + OB = AD.$

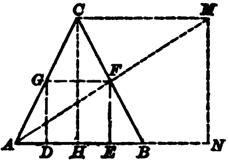
∴ $ABCD$ is the square required.

Ex. 279. To inscribe a square in a given triangle.

Let ABC be the given △. It is required to inscribe a square in △ ABC.

Construction. Draw the altitude CH. On HB produced construct a square $HNMC$ with CH as one side. Join AM, cutting BC at F. Construct the rectangle $DEFG$

DEFG is the square required.



Proof. In $\triangle ANM$,

EF is $\parallel$ to NM. (§ 100)

Whence $EF : NM = AF : AM$.

Also, it may be proved in $\triangle ACM$, that

$GF : CM = AF : AM$.

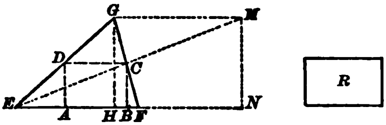
$\therefore EF : NM = GF : CM$.

But $NM = CM$ (Cons.). $\therefore EF = GF$.

$\therefore$ *DEFG* is the square required.

Ex. 280. To inscribe in a given triangle a rectangle similar to a given rectangle.

Let *EFG* be the given $\triangle$, and *R* the given rectangle. It is required to inscribe in *EFG* a rectangle similar to *R*.



Construction. Draw the altitude *GH*. On *HF* produced construct a rectangle similar to *R*. Join *EM*, cutting *FG* at *C*. Construct the rectangle *ABCD*. Then *ABCD* is the rectangle required.

Proof. In the $\triangle ENM$, BC is $\parallel$ to NM. (§ 100)

Whence $BC : NM = EC : EM$.

Also, it may be proved in $\triangle EGM$, that

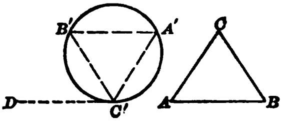
$DC : GM = EC : EM$.

$\therefore BC : NM = DC : GM$.

$\therefore$ *ABCD* is similar to *HNMG*, and also to *R*.

Ex. 281. To inscribe in a circle a triangle similar to a given triangle.

Let *O* be the given ⊙, and *ABC* the given $\triangle$. It is required to inscribe in *O* a $\triangle$ similar to *ABC*.



Construction. At any point in the circumference draw the tangent DC'. At C' draw a chord, making $\angle B'C'D$ equal to $\angle A$. At B' make the $\angle A'B'C'$ equal to $\angle B$. Join $A'C'$.

$\triangle A'B'C'$ is the $\triangle$ required.

Proof. $\angle A' = \angle B'C'D$ (being measured by $\frac{1}{2}$ the same arc).

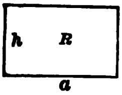
$\therefore \angle A' = \angle A$. $\angle B' = \angle B$. (Cons.)

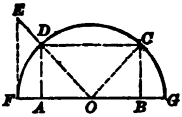
$\therefore$ $\triangle$ *ABC* and $A'B'C'$ are similar. (§ 322)

Ex. 282. To inscribe in a given semicircle a rectangle similar to a given rectangle.

Let R be the given rectangle, and $FDCG$ the given semicircle. It is required to inscribe in $FDCG$ a rectangle similar to R.

Construction. Find FE, a fourth proportional to $\frac{1}{2}a$, h, and the radius OF. (§ 352)





At F draw the tangent FE. Join OE, cutting the circumference at D. Construct the rectangle $ABCD$.

Then $ABCD$ is the rectangle required.

Proof. Draw OC. $OD = OC$. (radii)

$AD = BC$. (Cons.)

$\therefore \triangle OAD = \triangle OBC$. (§ 161)

$\therefore OA = OB$.

In $\triangle OFE$, AD is $\parallel$ to FE. (§ 100)

$\therefore OA : AD = OF : FE$.

But $\frac{1}{2}a : h = OF : FE$. (Cons.)

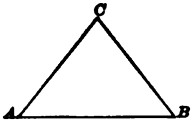
$\therefore OA : AD = \frac{1}{2}a : h$.

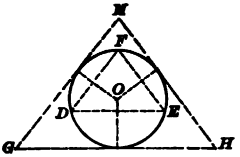
But $OA = \frac{1}{2}AB$.

$\therefore \frac{1}{2}AB : AD = \frac{1}{2}a : h$.

$\therefore ABCD$ is similar to R.

Ex. 283. To circumscribe about a circle a triangle similar to a given triangle.





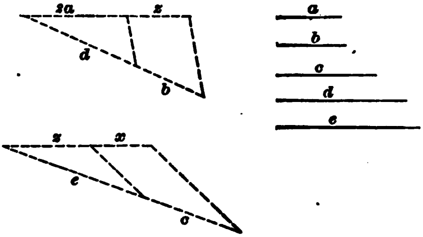
Let ABC be the given $\triangle$. It is required to circumscribe a $\triangle$ about the $\odot$ DEF similar to ABC.

Construction. Inscribe in the $\odot$ a $\triangle$ DEF similar to $\triangle$ ABC (Ex. 281). Draw radii $\perp$ to the sides of $\triangle$ DEF. At the extremities of these radii, draw the tangents GH, HM, and MG. Then $\triangle$ GHM is the $\triangle$ required

Proof. GH, HM, and MG are $\parallel$ to DE, EF, and FD, respectively (§ 240 and § 100). $\therefore$ ▲ GHM and DEF are similar. (§ 327)

$\therefore$ ▲ GHM and ABC are similar.

Ex. 284. To construct the expression, $x = \frac{2abc}{de}$; that is, $\frac{2ab}{d} \times \frac{c}{e}$.



Given the lines a, b, c, d, and e. It is required to construct the expression

$$x = \frac{2abc}{de}$$

Construction. Find z, a fourth proportional to d, $2a$, and b. (§ 352)

Find x, a fourth proportional to e, z, and c. (§ 352)

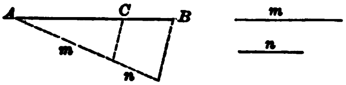
Then x is the line required.

Proof. $$d : 2a = b : z. \quad \therefore z = \frac{2ab}{d}.$$

$$e : z = c : x. \quad \therefore x = \frac{cz}{e}.$$

Substitute for z, $\frac{2ab}{d}$. Then $x = \frac{2abc}{de}$.

Ex. 285. To construct two straight lines, having given their sum and their ratio.

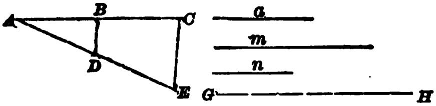


Let AB be the sum of two lines, and $m : n$ their ratio. It is required to find the lines.

Construction. Divide AB into segments proportional to m and n (§ 351). Then AC and CB are the lines required.

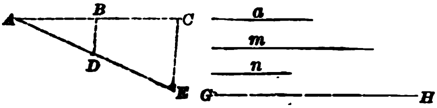
Proof. By construction $AC + CB = AB$, and $AC : CB = m : n$.

Ex. 286. To construct two straight lines, having given their difference and their ratio.



Let a equal the difference of two straight lines, and $m : n$ their ratio. It is required to find the lines.

Construction. Take AD equal to $m - n$. AB equal to n, and DE equal to a. Find BC, a fourth proportional to AD, AB, and DE (§ 352). Take GH equal to $BC + a$. Then BC and GH are the lines required.



Proof. By construction $AD : AB = DE : BC$.

Also, $AD = m - n$,

$AB = n$,

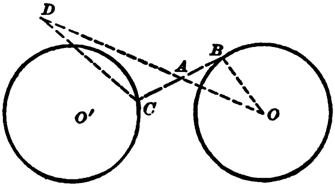
and $DE = GH - BC$.

Substituting these values, $\frac{m-n}{n} = \frac{GH - BC}{BC}$;

or $\frac{m}{n} - 1 = \frac{GH}{BC} - 1$. $\therefore GH : BC = m : n$.

Ex. 287. Having given two circles, with centres O and O', and a point A in their plane, to draw through the point A a straight line, meeting the circumferences at B and C, so that $AB : AC = 1 : 2$.

Let O and O' be the given $\odot$s, and A the given point. It is required to draw a line through A, meeting the circumferences at B and C, so that $AB : AC = 1 : 2$.



Construction. Draw OA, and prolong it to D, so that

$OA : AD = 1 : 2$. (§ 352)

With D as a centre, and with a radius double the radius of $\odot O$, cut the circumference of O' at C. Join DC. Draw $OB \parallel$ to CD. Join AB and AC. Then BAC is a straight line and $AB : AC = 1 : 2$.

Proof. In $\triangle$s ABO and ACD,

$\angle D = \angle O$. (§ 104)

$OA : AD = 1 : 2 = OB : DC$. (Cons.)

$\therefore \triangle$s ABO and ACD are similar. (§ 326)

$\therefore \angle OAB = \angle DAC$.

$\therefore BAC$ is a straight line.

Also, $AB : AC = OA : AD = 1 : 2$.

BOOK IV.

Ex. 288. Find the ratio of a rectangular lawn 72 yards by 49 yards to a grass turf 18 inches by 14 inches.

SOLUTION. 18 inches = $\frac{1}{2}$ yard, 14 inches = $\frac{7}{18}$ yard.

$$\text{Area of lawn : area of turf} = 72 \times 49 : \tfrac{1}{2} \times \tfrac{7}{18} \qquad (\S\ 362)$$
$$= 18144 : 1.$$

Ex. 289. Find the ratio of a rectangular courtyard $18\frac{1}{2}$ yards by $15\frac{1}{2}$ yards to a flagstone 31 inches by 18 inches.

SOLUTION. 31 inches = $\frac{31}{36}$ yard, 18 inches = $\frac{1}{2}$ yard.

$$\text{Area of courtyard : area of flagstone} = 18\tfrac{1}{2} \times 15\tfrac{1}{2} : \tfrac{31}{36} \times \tfrac{1}{2} \qquad (\S\ 362)$$
$$= 666 : 1.$$

Ex. 290. A square and a rectangle have the same perimeter, 100 yards. The length of the rectangle is 4 times its breadth. Compare their areas.

SOLUTION. x = the breadth of the rectangle.

$4x$ = the length of the rectangle.

$10x$ = the perimeter of the rectangle.

But 100 = the perimeter of the rectangle.

$\therefore 10x = 100,\ x = 10,\ 4x = 40.$

The length of the square = $\frac{1}{4}$ of 100 = 25.

$$\text{Area of rectangle : area of square} = 40 \times 10 : 25 \times 25 \qquad (\S\ 362)$$
$$= 16 : 25.$$

Ex. 291. On a certain map the linear scale is 1 inch to 5 miles. How many acres are represented on this map by a square, the perimeter of which is 1 inch?

SOLUTION. The length of the square is $\frac{1}{4}$ inch, and represents $\frac{5}{4}$ miles.

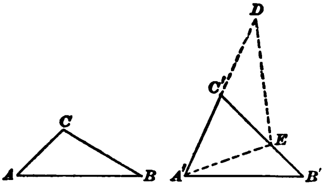
The area = $\frac{5}{4} \times \frac{5}{4} = \frac{25}{16}$ square miles.

$\frac{25}{16} \times 640 = 1000$ acres.

Ex. 292. The areas of two triangles which have an angle of the one supplementary to an angle of the other are to each other as the products of the sides including the supplementary angles.

Let the $\triangle$ ABC and $A'B'C'$ have the $\angle$ ACB and $A'C'B'$ supplements of each other.

To prove that $\triangle ABC : \triangle A'B'C' = CA \times CB : C'A' \times C'B'$.



Proof. Place $\triangle ABC$ in the position $C'DE$, so that $\angle$ ACB and $A'C'B'$ are adjacent. Draw $A'E$.

$A'C'D$ is a straight line. (§ 93)

Now $$\frac{\triangle C'DE}{\triangle A'C'E} = \frac{C'D}{C'A'}, \text{ and } \frac{\triangle A'C'E}{\triangle A'B'C'} = \frac{C'E}{C'B'}. \qquad (§\ 370)$$

Multiplying these equalities,

$$\frac{\triangle C'DE}{\triangle A'B'C'} = \frac{C'D \times C'E}{C'A' \times C'B'}.$$

Substituting, $\triangle ABC : \triangle A'B'C' = CA \times CB : C'A' \times C'B'$.

Exs. 293–295. See text-book.

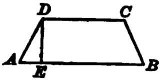
Ex. 296. If the perimeter of a rectangle is 72 feet, and the length is equal to twice the width, find the area.

Solution. $x =$ the breadth,
$2x =$ the length,
$6x =$ the perimeter.
But $72 =$ the perimeter.
$\therefore 6x = 72,\ x = 12,\ 2x = 24.$
Area $= 24 \times 12 = 288$ square feet.

Ex. 297. How many tiles 9 inches long and 4 inches wide will be required to pave a path 8 feet wide surrounding a rectangular court 120 feet long and 36 feet wide?

Solution. The area of the path on the longer side $= 136 \times 8 = 1088$.
The area of the path on the shorter side $= 36 \times 8 = 288$.
Area of the path $= 1088 \times 2 + 288 \times 2 = 2752$ square feet.
9 inches $= \frac{3}{4}$ foot, 4 inches $= \frac{1}{3}$ foot.
Area of a tile $= \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ square foot.
Number of tiles $= 2752 \div \frac{1}{4} = 11{,}008$.

Ex. 298. The bases of a trapezoid are 16 feet and 10 feet; each leg is equal to 5 feet. Find the area of the trapezoid.



Let $ABCD$ be a trapezoid, and DE be its altitude; $AB = 16$, $CD = 10$, $AD = 5 = BC$.

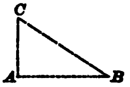
Find the area of the trapezoid.

Solution. Since $AD = BC$, $AE = \frac{1}{2}(AB - CD) = \frac{1}{2}(16 - 10) = 3$.

$$DE = \sqrt{\overline{AD}^2 - \overline{AE}^2} = \sqrt{5^2 - 3^2} = 4. \qquad (§\ 339)$$

$$\text{Area of the trapezoid } ABCD = DE \times \tfrac{1}{2}(AB + CD)$$
$$= 4 \times \tfrac{1}{2}(16 + 10)$$
$$= 52 \text{ square feet.} \qquad (§\ 371)$$

Ex. 299. Construct a square equivalent to the sum of two squares whose sides are 3 inches and 4 inches.



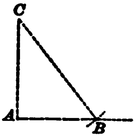
Solution. Construct the rt. ∠ A.

Take $AB = 4$, and $AC = 3$. Draw BC.

BC is a side of the square required. (§ 381)

$$BC = \sqrt{\overline{AB}^2 + \overline{AC}^2} = \sqrt{4^2 + 3^2} = 5 \text{ inches.} \qquad (§\ 338)$$

Ex. 300. Construct a square equivalent to the difference of two squares whose sides are $2\frac{1}{2}$ inches and 2 inches.



Solution. Construct the rt. ∠ A. Take $AC = 2$.

From C as a centre, with a radius equal to $2\frac{1}{2}$, describe an arc cutting AB at B. Draw BC.

AB is a side of the square required. (§ 382)

$$AB = \sqrt{\overline{BC}^2 - \overline{AC}^2} = \sqrt{(\tfrac{5}{2})^2 - 2^2} = 1\tfrac{1}{2} \text{ inches.} \qquad (§\ 339)$$

Ex. 301. Find the side of a square equivalent to the sum of two squares whose sides are 24 feet and 32 feet.

Solution. x = a side of the square required.

$$x = \sqrt{24^2 + 32^2} = 40 \text{ feet.} \qquad (§\ 381)$$

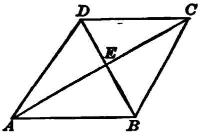
Ex. 302. Find the side of a square equivalent to the difference of two squares whose sides are 24 feet and 40 feet.

Solution. x = a side of the square required.

$$x = \sqrt{40^2 - 24^2} = 32 \text{ feet.} \qquad (§\ 382)$$

Ex. 303. A rhombus contains 100 square feet, and the length of one diagonal is 10 feet. Find the length of the other diagonal.

Let $ABCD$ be a rhombus containing 100 square feet, and let $BD = 10$ feet. Find the length of AC.



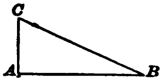
Solution. AC is $\perp$ to BD. (Ex. 28)

$\therefore$ the area of the rhombus $ABCD = \frac{1}{2}(AC \times BD)$

[since $\triangle ADC + \triangle ABC = \frac{1}{2}(AC \times DE)$
$+ \frac{1}{2}(AC \times BE)$].

Substituting, $100 = \frac{1}{2}(AC \times 10)$. $\therefore 5AC = 100$; $AC = 20$ feet.

Ex. 304. Find the area of a right triangle if the length of the hypotenuse is 17 feet, and the length of one leg is 8 feet.



In the rt. $\triangle ABC$,

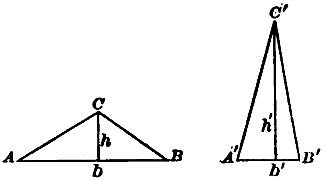
$AC = 8$ feet, and $BC = 17$ feet.

Find the length of AB.

Solution. $AB = \sqrt{\overline{BC}^2 - \overline{AC}^2} = \sqrt{17^2 - 8^2} = 15$. (§ 339)

Area of $\triangle ABC = \frac{1}{2}(AB \times AC) = \frac{1}{2}(15 \times 8) = 60$ square feet.

Ex. 305. Compare the altitudes of two equivalent triangles, if the base of one is three times that of the other.



Let the equivalent $\triangle$s ABC and $A'B'C'$ have the bases b and b', and the altitudes h and h', respectively. $b = 3b'$.

Find the value of the ratio $h : h'$.

Solution. $\triangle ABC : \triangle A'B'C' = b \times h : b' \times h'$. (§ 370)

But $\triangle ABC \Leftrightarrow \triangle A'B'C'$. (Hyp.)

$\therefore b \times h = b' \times h'$; and $b' : b = h : h'$. (§ 297)

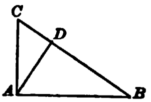
But $b' : b = 1 : 3$; hence $h : h' = 1 : 3$.

Ex. 306. The bases of a trapezoid are 8 feet and 10 feet, and the altitude is 6 feet. Find the base of an equivalent rectangle having an equal altitude.

Solution. $x =$ the base of the rectangle.
The area of the trapezoid $= 6 \times \frac{1}{2}(8 + 10)$.
The area of the rectangle $= 6x$.
$\therefore 6x = 6 \times \frac{1}{2}(8 + 10)$; and $x = 9$ feet.

Exs. 307–309. See text-book.

Ex. 310. In a right triangle the product of the legs is equal to the product of the hypotenuse and the perpendicular drawn to the hypotenuse from the vertex of the right angle.



Let ABC be a rt. $\triangle$, and AD a $\perp$ to the hypotenuse.

Then $AB \times AC = BC \times AD.$

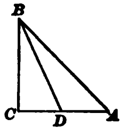
Proof. The area of $\triangle ABC = \frac{1}{2}(AB \times AC).$

The area of $\triangle ABC = \frac{1}{2}(BC \times AD).$ (§ 368)

$\therefore AB \times AC = BC \times AD.$ (Ax. 6)

Ex. 311. If ABC is a right triangle, C the vertex of the right angle, BD a line cutting AC in D, then

$$\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2.$$



In the rt. $\triangle ABC$, let BD be drawn, cutting AC in D.

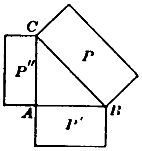
Then $\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2.$

Proof. $\overline{BD}^2 = \overline{BC}^2 + \overline{DC}^2$ and $\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2.$ (§ 338)

Add these equalities; then

$$\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2.$$

Ex. 312. Upon the sides of a right triangle as homologous sides three similar polygons are constructed. Prove that the polygon upon the hypotenuse is equivalent to the sum of the polygons upon the legs.



Let ABC be a rt. $\triangle$.

Let P, P', and P'' be three similar polygons constructed upon BC, AB, and AC, as homologous sides.

Then $P \Leftrightarrow P' + P''.$

Proof. $\dfrac{P'}{P} = \dfrac{\overline{AB}^2}{\overline{BC}^2}$ and $\dfrac{P''}{P} = \dfrac{\overline{AC}^2}{\overline{BC}^2}$ (§ 376)

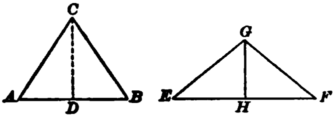
Add these equalities; then

$$\frac{P' + P''}{P} = \frac{\overline{AB}^2 + \overline{AC}^2}{\overline{BC}^2}.$$

But $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2.$ (§ 338)

$$\therefore P \Leftrightarrow P' + P''.$$

Ex. 313. Two isosceles triangles are equivalent if their legs are equal each to each, and the altitude of one is equal to half the base of the other.



In the isosceles ▲ ABC and EFG, let AC and BC equal EG and FG, and let GH, the altitude of EFG, equal $\frac{1}{2}AB$.

Then $\triangle ABC \Leftrightarrow \triangle EFG$.

PROOF. Draw $CD \perp$ to AB; then $AD = \frac{1}{2}AB$. (Ex. 16)

In the rt. ▲ ADC and EHG,

$AC = EG$ and $GH = AD$. (Hyp.)

$\therefore \triangle ADC = \triangle EHG$. (§ 161)

In the same way it may be proved that

$\triangle BDC = \triangle FHG$.

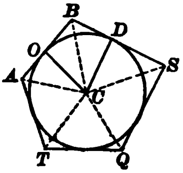
Add these equalities; then

$\triangle ADC + \triangle BDC \Leftrightarrow \triangle EHG + \triangle FHG$.

Or $\triangle ABC \Leftrightarrow \triangle EFG$.

Ex. 314. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.

Let P represent the perimeter of the circumscribed polygon $ABSQT$, and OC be the radius of the inscribed circle.



Then the area of the polygon

$ABSQT = \frac{1}{2}(P \times OC)$.

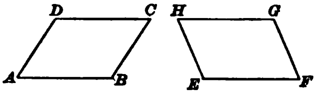
PROOF. Draw CA, CB, CS, CQ, and CT.

The area of $\triangle ACB = \frac{1}{2}(AB \times OC)$. (§ 368)

In the same way it may be proved that the area of each $\triangle$ equals half the product of OC and its base.

Therefore the area of the polygon, which equals the sum of the areas of the ▲, is equal to $\frac{1}{2}(P \times OC)$.

Ex. 315. Two parallelograms are equal if two adjacent sides of the one are equal respectively to two adjacent sides of the other, and the included angles are supplementary.



Let $ABCD$ and $EFGH$ be two ▱, with AB and AD equal respectively to EF and EH, and the $\angle$ A and E supplementary. Then ▱ $ABCD$ = ▱ $EFGH$.

Proof. $\angle F + \angle E = 2$ rt. $\angle$s. (§ 109)

But $\angle A + \angle E = 2$ rt. $\angle$s. (Hyp.)

$\therefore \angle F = \angle A.$

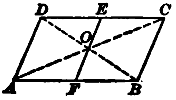
Also, $FG = EH.$ (§ 179)

But $EH = AD.$ (Hyp.)

$\therefore FG = AD.$ (Ax. 1)

$\therefore \square ABCD = \square EFGH.$ (§ 185)

Ex. 316. Every straight line drawn through the centre of a parallelogram divides it into two equal parts.



Let *ABCD* be the given $\square$, *O* its centre, and *EF* any line drawn through *O*.

Then *EF* divides the $\square$ *ABCD* into two equal parts.

Proof. Draw the diagonals *AC* and *BD*; *AC* and *BD* intersect at *O*. (§ 184)

In the $\triangle$s *AOF* and *EOC*, $AO = OC,$ (§ 184)

$\angle OAF = \angle OCE,$ (§ 104)

and $\angle AOF = \angle EOC.$ (§ 95)

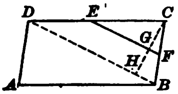
$\therefore \triangle AOF = \triangle EOC.$ (§ 147)

In the same way it may be proved that

$\triangle FOB = \triangle DOE$ and $\triangle AOD = \triangle BOC.$

Therefore the quadrilateral $AFED =$ the quadrilateral *FBCE* (being composed of equal $\triangle$s similarly placed).

Ex. 317. If the middle points of two adjacent sides of a parallelogram are joined, a triangle is formed which is equivalent to one-eighth of the entire parallelogram.



Let *ABCD* be a $\square$, and let *EF* join the middle points of *CD* and *BC*.

Then $\triangle EFC \Leftrightarrow \frac{1}{8} \square ABCD.$

Proof. Draw the diagonal *BD*, and $CH \perp$ to *BD*.

FE is $\parallel$ to *BD* and $= \frac{1}{2} BD.$ (§ 189)

$CG = \frac{1}{2} CH.$ (§ 188)

$\triangle DBC : \triangle EFC = BD \times CH : EF \times CG.$ (§ 370)

But $EF = \frac{1}{2} BD$ and $CG = \frac{1}{2} CH.$

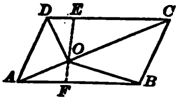
$\therefore \triangle DBC : \triangle EFC = BD \times CH : \frac{1}{2} BD \times \frac{1}{2} CH = 4 : 1.$

But $\triangle BCD \Leftrightarrow \frac{1}{2} \square ABCD.$ (§ 178)

$\therefore \triangle EFC (\Leftrightarrow \frac{1}{4} \triangle BCD) \Leftrightarrow \frac{1}{8} \square ABCD.$

Ex. 318. If any point within a parallelogram is joined to the four vertices, the sum of either pair of triangles having parallel bases is equivalent to one-half the parallelogram.

Let O be any point within the $\square$ $ABCD$, and let OA, OB, OC, and OD be drawn.



Then $\triangle AOB + \triangle DOC \Leftrightarrow \frac{1}{2} \square ABCD$.

PROOF. Draw $EOF \perp$ to AB.

The area of $\triangle AOB = \frac{1}{2}(AB \times OF)$,

and the area of $\triangle DOC = \frac{1}{2}(DC \times OE)$. (§ 368)

Add these equalities, substituting AB for DC.

Then the area of $\triangle AOB$ + the area of $\triangle DOC = \frac{1}{2}(AB \times EF)$.

But the area of $\square ABCD = AB \times EF$. (§ 365)

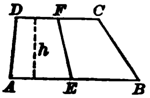
$$\therefore \triangle AOB + \triangle DOC \Leftrightarrow \tfrac{1}{2} \square ABCD.$$

In the same way it may be proved that

$$\triangle AOD + \triangle BOC \Leftrightarrow \tfrac{1}{2} \square ABCD.$$

Ex. 319. The line which joins the middle points of the bases of a trapezoid divides the trapezoid into two equivalent parts.

Let $ABCD$ be a trapezoid, and EF a line joining the middle points of the bases AB and CD.



Then the trapezoid $AEFD \Leftrightarrow$ the trapezoid $EBCF$.

PROOF. Draw h, the altitude of the trapezoid $ABCD$.

The area of the trapezoid $AEFD = \frac{1}{2} h (AE + DF)$,

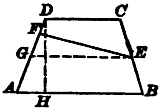
and the area of the trapezoid $EBCF = \frac{1}{2} h (EB + FC)$. (§ 371)

But $AE + DF = EB + FC$. (Hyp.)

$\therefore$ the trapezoid $AEFD \Leftrightarrow$ the trapezoid $EBCF$.

Ex. 320. The area of a trapezoid is equal to the product of one of the legs and the distance from this leg to the middle point of the other leg.

In the trapezoid $ABCD$, let EF be drawn from the middle point of $BC \perp$ to AD.



Then the area of the trapezoid $ABCD = AD \times EF$.

PROOF. Draw the median EG and the altitude DH.

In the rt. $\triangle$s ADH and EFG,

$\angle EGF = \angle A$ (§ 106). $\therefore$ $\triangle$s ADH and EFG are similar. (§ 323)

Whence, $AD : EG = DH : EF.$

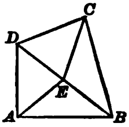
$\therefore AD \times EF = EG \times DH.$

But the area of the trapezoid $ABCD = EG \times DH.$ (§ 372)

$\therefore$ the area of the trapezoid $ABCD = AD \times EF.$ (Ax. 1)

Ex. 321. The lines joining the middle point of the diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.

In the quadrilateral *ABCD*, let *E* be the middle point of the diagonal *BD*, and let *AE* and *CE* be drawn.



Then the quadrilateral $AECD \Leftrightarrow$ the quadrilateral *ABCE*.

Proof. $\triangle ABE \Leftrightarrow \triangle AED$, and $\triangle bEC \Leftrightarrow \triangle CED.$ (§ 369)

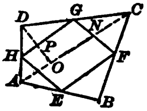
Add these equalities; then

$$\triangle ABE + \triangle BEC \Leftrightarrow \triangle AED + \triangle CED.$$

Or the quadrilateral $AECD \Leftrightarrow$ the quadrilateral *ABCE*.

Ex. 322. The figure whose vertices are the middle points of the sides of any quadrilateral is equivalent to one-half of the quadrilateral.

In the quadrilateral *ABCD*, let *EF*, *FG*, *GH*, and *HE* join the middle points of *AB*, *BC*, *CD*, and *DA*.



Then the $\text{▱}\, EFGH \Leftrightarrow \frac{1}{2}$ the quadrilateral *ABCD*.

Proof. Draw the diagonal *AC*, and $DO \perp$ to *AC*.

HG is $\parallel$ to AC and $= \frac{1}{2} AC.$ (§ 189)

$PO = \frac{1}{2} DO.$ (§ 188)

The area of $\text{▱}\, HN = HG \times PO.$ (§ 365)

The area of $\triangle ADC = \frac{1}{2}(AC \times DO).$ (§ 368)

But $HG = \frac{1}{2} AC$ and $PO = \frac{1}{2} DO.$

$\therefore \text{▱}\, HN \Leftrightarrow \frac{1}{2} \triangle ADC.$

In the same way it may be proved that

$$\text{▱}\, EN \Leftrightarrow \tfrac{1}{2} \triangle ABC.$$

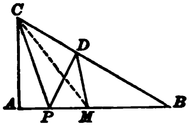
Add these equalities; then

$$\text{▱}\, HN + \text{▱}\, EN \Leftrightarrow \tfrac{1}{2}(\triangle ADC + \triangle ABC).$$

Or $\text{▱}\, EFGH \Leftrightarrow \frac{1}{2}$ the quadrilateral *ABCD*.

Ex. 323. ABC is a triangle, M the middle point of AB, P any point in AB between A and M. If MD is drawn parallel to PC, and meeting BC at D, the triangle BPD is equivalent to one-half the triangle ABC.

In the $\triangle\, ABC$, let M be the middle point of AB, and P any point of AM. Let MD be drawn $\parallel$ to PC. Join PD.



Then $\triangle\, BPD \Leftrightarrow \frac{1}{2} \triangle\, ABC$.

PROOF. Draw MC.

$BM = \frac{1}{2} AB$. (Hyp.)

$\therefore \triangle\, BMC \Leftrightarrow \frac{1}{2} \triangle\, ABC$. (§ 370)

In $\triangle$ BMC and BPD, $\triangle\, BMD$ is common.

$\triangle\, MPD \Leftrightarrow \triangle\, MDC$. (§ 369)

$\therefore \triangle\, BMC \Leftrightarrow \triangle\, BPD$. (Ax. 2)

$\therefore \triangle\, BPD \Leftrightarrow \frac{1}{2} \triangle\, ABC$. (Ax. 1)

Ex. 324. Find the area of a rhombus, if the sum of its diagonals is 12 feet, and their ratio 3 : 5.

SOLUTION. Let $5x =$ one diagonal, $3x =$ the other.

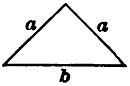
Then $5x + 3x = 12$. $\therefore x = 1\frac{1}{2}$.

$5x = 7\frac{1}{2}$, $3x = 4\frac{1}{2}$.

The area of a rhombus equals $\frac{1}{2}$ the product of the diagonals (since the diagonals are $\perp$ to each other).

Therefore the area $= \frac{1}{2}(7\frac{1}{2} \times 4\frac{1}{2}) = 16\frac{7}{8}$ square feet.

Ex. 325. Find the area of an isosceles right triangle if the hypotenuse is 20 feet.



In the isosceles rt. $\triangle$ let the hypotenuse b equal 20 feet. Find the area of the $\triangle$.

SOLUTION. $2a^2 = b^2$ (§ 338). $\therefore a = b\sqrt{\frac{1}{2}} = 20\sqrt{\frac{1}{2}}$.

The area of the $\triangle = \frac{1}{2}a^2$ (§ 368) $= \frac{1}{2}(20\sqrt{\frac{1}{2}})^2 = 100$ square feet.

Ex. 326. In a right triangle, the hypotenuse is 13 feet, one leg is 5 feet. Find the area.

SOLUTION. $x =$ the length of the other leg.

$\therefore x = \sqrt{13^2 - 5^2}$ (§ 339)

$= 12$.

The area $= \frac{1}{2}(5 \times 12)$ (§ 368)

$= 30$ square feet.

Ex. 327. Find the area of an isosceles triangle if the base $= b$, and leg $= c$.

SOLUTION. Let h be the altitude.

Then
$$h = \sqrt{c^2 - (\tfrac{1}{2}b)^2} \qquad (\S\ 339)$$
$$= \sqrt{\frac{4c^2 - b^2}{4}} = \tfrac{1}{2}\sqrt{(2c+b)(2c-b)}.$$
$$\text{The area} = \tfrac{1}{2}(h \times b) \qquad (\S\ 368)$$
$$= \tfrac{1}{4}b\sqrt{(2c+b)(2c-b)}.$$

Ex. 328. Find the area of an equilateral triangle if one side $=$ 8 feet.

SOLUTION.
$$\text{The area} = \frac{a^2\sqrt{3}}{4} \qquad (\text{Ex. } 307)$$
$$= \frac{8^2\sqrt{3}}{4} = 16\sqrt{3} = 27.712 \text{ square feet.}$$

Ex. 329. Find the area of an equilateral triangle if the altitude $= h$.

SOLUTION.
$$h = \frac{a}{2}\sqrt{3}. \qquad (\text{Ex. } 307)$$
$$\therefore a = \frac{2h}{\sqrt{3}} = \tfrac{2}{3}h\sqrt{3}.$$
$$\text{The area} = \tfrac{1}{2}(h \times a) \qquad (\S\ 368)$$
$$= \tfrac{1}{2}(h \times \tfrac{2}{3}h\sqrt{3}) = \frac{h^2}{3}\sqrt{3}.$$

Ex. 330. A house is 40 feet long, 30 feet wide, 25 feet high to the eaves, and 35 feet high to the ridge-pole. Find the number of square feet in its entire exterior surface.

SOLUTION. The area of 4 sides $= 2(40 + 30)\,25 = 3500$ square feet.

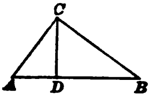
The area of the gable ends $= 2 \times \tfrac{1}{2}(10 \times 30) = 300$ square feet.

The length of the rafters $= \sqrt{10^2 + 15^2} = 18.027$.

The area of the roof $= 2 \times 18.027 \times 40 = 1442.16$ sq. ft.

The entire area $= 3500 + 300 + 1442.16 = 5242.16$ sq. ft.

Ex. 331. The sides of a right triangle are as 3 : 4 : 5. The altitude upon the hypotenuse is 12 feet. Find the area.



In the rt. △ ABC, let DC, the altitude upon the hypotenuse, equal 12 feet, and let

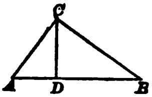
$$AC : BC : AB = 3 : 4 : 5.$$

Find the area of △ ABC.

SOLUTION. $AB = 5x,\ AC = 3x,\ BC = 4x.$

$$5x : 3x = 3x : AD. \qquad (\S\ 334,\ \text{II.})$$

$$\therefore AD = \frac{9x}{5}$$



$$\therefore DB = 5x - \frac{9x}{5} = \frac{16x}{5}.$$

$$\frac{9x}{5} : 12 = 12 : \frac{16x}{5}. \quad (\S\ 334,\ \text{I.})$$

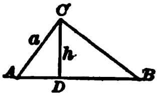
$$\therefore \frac{144x^2}{25} = 144,\quad x^2 = 25,\quad x = 5.$$

$$\therefore AB = 25.$$

The area of $\triangle ABC = \frac{1}{2}(12 \times 25)$ (§ 368)

$$= 150 \text{ square feet.}$$

Ex. 332. Find the area of a right triangle if one leg $= a$, and the altitude upon the hypotenuse $= h$.



SOLUTION. $AD = \sqrt{a^2 - h^2}.$ (§ 339)

$$AB : a = a : AD$$

$$= a : \sqrt{a^2 - h^2} \quad (\S\ 334,\ \text{II.})$$

$$\therefore AB = \frac{a^2}{\sqrt{a^2 - h^2}}.$$

The area of $\triangle ABC = \frac{1}{2}(h \times AB)$ (§ 368)

$$= \frac{1}{2}\left(h \times \frac{a^2}{\sqrt{a^2 - h^2}}\right) = \frac{a^2 h}{2\sqrt{(a+h)(a-h)}}.$$

Ex. 333. Find the area of a triangle if the lengths of the sides are 104 feet, 111 feet, and 175 feet.

SOLUTION. The area $= \sqrt{s(s-a)(s-b)(s-c)}$ (Ex. 308)

$$= \sqrt{195(195-104)(195-111)(195-175)}$$

$$= 5460 \text{ square feet.}$$

Ex. 334. The area of a trapezoid is 700 square feet. The bases are 30 feet and 40 feet respectively. Find the distance between the bases.

SOLUTION. Let h represent the altitude.

The area of the trapezoid $= \frac{1}{2}(40 + 30)h.$ (§ 371)

$$\therefore 700 = \frac{1}{2}(40 + 30)h = 35h.$$

$$\therefore h = 20 \text{ feet.}$$

Ex. 335. $ABCD$ is a trapezium; $AB = 87$ feet, $BC = 119$ feet, $CD = 41$ feet, $DA = 169$ feet, $AC = 200$ feet. Find the area.

Solution. The area of $\triangle\ ABC$

$$= \sqrt{203\,(203 - 87)(203 - 119)(203 - 200)}$$

$$= 2436 \text{ square feet;}$$

and the area of $\triangle\ ADC$

$$= \sqrt{205\,(205 - 200)\,(205 - 41)(205 - 169)} \qquad \text{(Ex. 308)}$$

$$= 2460 \text{ square feet.}$$

Therefore the area of $ABCD = 2436 + 2460 = 4896$ square feet.

Ex. 336. What is the area of a quadrilateral circumscribed about a circle whose radius is 25 feet, if the perimeter of the quadrilateral is 400 feet? What is the area of a hexagon having an equal perimeter and circumscribed about the same circle?

Solution. By Ex. 314, the area of the quadrilateral

$$= \tfrac{1}{2}\,(400 \times 25) = 5000 \text{ square feet.}$$

The hexagon has the same area.

Ex. 337. The base of a triangle is 15 feet, and its altitude is 8 feet. Find the perimeter of an equivalent rhombus if the altitude is 6 feet.

Solution. The area of the $\triangle = \tfrac{1}{2}\,(8 \times 15)$ (§ 368)

$= 60$ square feet.

One side of the rhombus $= 60 \div 6 = 10$ feet.

The perimeter $= 4 \times 10 = 40$ feet.

Ex. 338. Upon the diagonal of a rectangle 24 feet by 10 feet a triangle equivalent to the rectangle is constructed. What is its altitude?

Solution. The area of the rectangle $= 24 \times 10 = 240$ square feet.

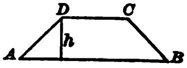
The diagonal of the rectangle $= \sqrt{24^2 + 10^2} = 26$ feet.

Let h represent the altitude of the $\triangle$.

Then the area of the $\triangle$ $= \tfrac{1}{2}\,(h \times 26)$. (§ 368)

$$\therefore\ 240 = \tfrac{1}{2}\,(h \times 26) = 13\,h. \quad \therefore\ h = 18\tfrac{6}{13} \text{ feet.}$$

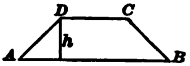
Ex. 339. Find the side of a square equivalent to a trapezoid whose bases are 56 feet and 44 feet, and each leg is 10 feet.



In the trapezoid $ABCD$, $AB = 56$, $DC = 44$, and $AD = BC = 10$.

Find the side of a square equivalent to the trapezoid $ABCD$.

SOLUTION. Let h represent the altitude of the trapezoid $ABCD$, and x a side of the square.



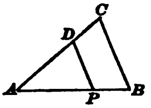
Since the trapezoid $ABCD$ is isosceles,

$$h = \sqrt{\overline{AD}^2 - (\tfrac{1}{2}AB - \tfrac{1}{2}DC)^2}$$
$$= \sqrt{10^2 - 6^2} = 8 \text{ feet}.$$

The area of the trapezoid $ABCD = \frac{1}{2}(56 + 44)\,8$ (§ 371)

$$= 400 \text{ square feet}.$$
$$\therefore x^2 = 400,\ \ x = 20 \text{ feet}.$$

Ex. 340. Through a point P in the side AB of a triangle ABC, a line is drawn parallel to BC, and so as to divide the triangle into two equivalent parts. Find the value of AP in terms of AB.

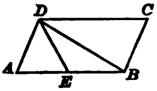


In the $\triangle ABC$, let PD be drawn $\parallel$ to BC, dividing $\triangle ABC$ into two equivalent parts. Find the value of AP in terms of AB.

SOLUTION. $\triangle APD : \triangle ABC = \overline{AP}^2 : \overline{AB}^2$ (§ 375)

$$= 1 : 2.$$
$$\therefore \overline{AP}^2 = \tfrac{1}{2}\overline{AB}^2.$$
$$\therefore AP = AB\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}AB\sqrt{2}.$$

Ex. 341. What part of a parallelogram is the triangle cut off by a line drawn from one vertex to the middle point of one of the opposite sides?



In the $\square\, ABCD$, let DE be drawn to the middle point of AB. What part of $\square\, ABCD$ is the $\triangle ADE$?

SOLUTION. $\triangle ADE \Bumpeq \frac{1}{2}\triangle ADB.$ (§ 370)

But $\triangle ADB \Bumpeq \frac{1}{2}\square\, ABCD.$ (§ 178)

$$\therefore \triangle ADE \Bumpeq \tfrac{1}{4}\square\, ABCD.$$

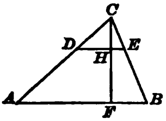
Ex. 342. In two similar polygons, two homologous sides are 15 feet and 25 feet. The area of the first polygon is 450 square feet. Find the area of the other polygon.

SOLUTION. Let S represent the area of the first polygon, and S' the area of the second.

Then $S : S' = 15^2 : 25^2.$ (§ 376)

$$450 : S' = 15^2 : 25^2.$$
$$\therefore S' = 1250 \text{ square feet}.$$

Ex. 343. The base of a triangle is 32 feet, its altitude 20 feet. What is the area of the triangle cut off by drawing a line parallel to the base and at a distance of 15 feet from the base?



In the $\triangle ABC$, $AB = 32$, $CF = 20$.

Let DE be drawn ∥ to AB, cutting off $HF = 15$. Find the area of $\triangle DEC$.

Solution. The area of $\triangle ABC = \frac{1}{2}(32 \times 20)$

$$= 320 \text{ square feet.} \qquad (§\ 368)$$

$$CH = 20 - 15 = 5 \text{ feet.}$$

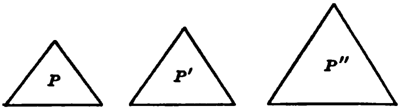
$$\triangle DEC : \triangle ABC = \overline{CH}^2 : \overline{CF}^2 \qquad (§\ 377)$$

$$= 5^2 : 20^2 = 1 : 16.$$

Therefore the area of $\triangle DEC = \frac{1}{16} \times 320 = 20$ square feet.

Ex. 344. The sides of two equilateral triangles are 3 feet and 4 feet. Find the side of an equilateral triangle equivalent to their sum.

Let P and P' be two equilateral $\triangle$ whose sides are 3 feet and 4 feet respectively. Find the side of an equilateral $\triangle$ equivalent to their sum.



Solution. Let P'' be the required equilateral $\triangle$, and let x represent one side of P''.

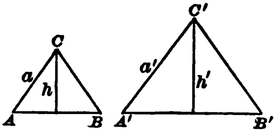
$$\frac{P}{P''} = \frac{3^2}{x^2} \text{ and } \frac{P'}{P''} = \frac{4^2}{x^2}. \qquad (§\ 375)$$

Add these equalities; then

$$\frac{P + P'}{P''} = \frac{3^2 + 4^2}{x^2}.$$

But $P + P' \Leftrightarrow P''$. $\therefore 3^2 + 4^2 = x^2$. $\therefore x = 5$ feet.

Ex. 345. If the side of one equilateral triangle is equal to the altitude of another, what is the ratio of their areas?



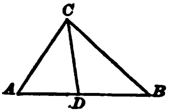
In the equilateral $\triangle$ ABC and $A'B'C'$, let a, a side of ABC, equal h', the altitude of $A'B'C'$. Find the ratio of the areas of $\triangle$ ABC and $A'B'C'$.

Solution. $\triangle ABC : \triangle A'B'C' = h^2 : h'^2$ (§ 377)

But $h^2 = \frac{3a^2}{4}$ (Ex. 307), and $h'^2 = a^2$. (Hyp.)

Substituting, $\triangle ABC : \triangle A'B'C' = \frac{3a^2}{4} : a^2 = 3 : 4.$

Ex. 346. The sides of a triangle are 10 feet, 17 feet, and 21 feet. Find the areas of the parts into which the triangle is divided by bisecting the angle formed by the first two sides.



In the $\triangle ABC$, $AC = 10$ feet, $BC = 17$ feet, and $AB = 21$ feet. Let CD bisect the $\angle C$.

Find the areas of $\triangle$s ADC and BDC.

SOLUTION. The area of $\triangle ABC$

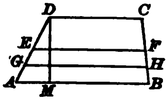
$$= \sqrt{24(24-10)(24-17)(24-21)} \quad \text{(Ex. 308)}$$
$$= 84 \text{ square feet.}$$

$$\triangle ADC : \triangle BDC = 10 : 17. \quad (\S\S\ 313, 370)$$

$$\therefore \triangle ADC \bumpeq \tfrac{10}{27} \text{ of } \triangle ABC = \tfrac{10}{27} \text{ of } 84 = 31\tfrac{1}{9} \text{ square feet,}$$

and $$\triangle BDC \bumpeq \tfrac{17}{27} \text{ of } \triangle ABC = \tfrac{17}{27} \text{ of } 84 = 52\tfrac{8}{9} \text{ square feet.}$$

Ex. 347. In a trapezoid one base is 10 feet, the altitude is 4 feet, the area is 32 square feet. Find the length of a line drawn between the legs parallel to the base and distant 1 foot from it.



In the trapezoid $ABCD$, $AB = 10$ feet, $DM = 4$ feet, and the area $= 32$ square feet.

Let EF be the median and GH be a line parallel to AB and 1 foot distant from AB. Find the length of GH.

SOLUTION. The area of the trapezoid $ABCD = EF \times DM$. (§ 372)

$$\therefore 32 = EF \times 4; \text{ and hence } EF = 8 \text{ feet.}$$

$$EF \text{ bisects } DM. \quad (\S\ 187)$$

Therefore in the trapezoid $ABFE$,

$$GH \text{ is the median and } = \tfrac{1}{2}(10 + 8) = 9 \text{ feet.}$$

Ex. 348. If the altitude h of a triangle is increased by a length m, how much must be taken from the base a in order that the area may remain the same?

SOLUTION. $x =$ the amount taken from the base.

$h + m =$ the increased altitude.

$a - x =$ the diminished base.

$\frac{1}{2}(h \times a) =$ the area of the first $\triangle$,

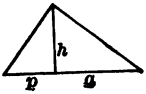
$\frac{1}{2}(h + m)(a - x) =$ the area of the second $\triangle$.

$$\therefore \tfrac{1}{2}(h \times a) = \tfrac{1}{2}(h + m)(a - x).$$
$$ha = ha - hx + am - mx.$$

Transposing, $hx + mx = am$.

$$\therefore x = \frac{am}{h + m}.$$

Ex. 349. Find the area of a right triangle, having given the segments p, q, into which the hypotenuse is divided by a perpendicular drawn to the hypoten The proof is the of the right angle.



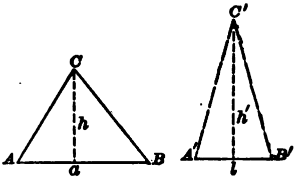
Solution q. (§ 334, I.)

To transform a g$\overline{\times q}$.

The ving the base equal $+ q)h = \frac{1}{2}(p + q)\sqrt{p \times q}$.

h the given △

Ex. 350. To construct a triangle equivalent to a given triangle, and having one side equal to a given length l.



Let ABC be the given △, a its base, and l the given line. It is required to construct a △ on l as a base equivalent to △ ABC.

Construction. Draw h, the altitude of △ ABC. Find h', a fourth proportional to l, a, and h. (§ 352)

On l as a base, with an altitude h', construct △ $A'B'C'$. Then △ $A'B'C'$ is the △ required.

Proof. $l : a = h : h'$; and hence $l \times h' = a \times h$.

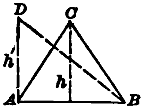
But $l \times h' =$ twice the area of △ $A'B'C'$,

and $a \times h =$ twice the area of △ ABC. (§ 368)

$\therefore$ △ $A'B'C' \backsimeq$ △ ABC.

Ex. 351. To transform a triangle into an equivalent right triangle.

Let ABC be the given △. It is required to transform △ ABC into an equivalent rt. △.

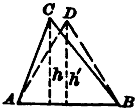


Construction. Draw h, the altitude of △ ABC. At the extremity of ABC erect a ⊥ $AD = h$ (§ 271). Join DB. △ ABD is the △ required.

Proof. △ $ABD \backsimeq$ △ ABC. (§ 369)

Ex. 352. To transform a triangle into an equivalent isosceles triangle.

Let ABC be the given △. It is required to transform △ ABC into an equivalent isosceles △.



Construction. Draw h, the altitude of △ ABC. At the middle point of AB, erect a ⊥ $h' = h$ (§ 273). Join AD and BD. △ ABD is the △ required.

Proof. $AD = BD$ (§ 116); and hence △ ABD is isosceles.

△ $ABD \backsimeq$ △ ABC. (§ 369)

Ex. 353. To transform a triangle ABC into an equivalent triangle, having one side equal to a given length l, and one angle equal to angle BAC.

Let ABC be the given △, and l the given length. It is required to transform △ ABC into an equivalent △ having one side equal to l.

17 feet, and

CONSTRUCTION. Upon AB take $AD = l$. Join CD. Draw $BE \parallel$ to DC, meeting AC produced at E. Join DE.

△ ADE is the △ required.

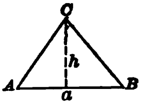
PROOF. In ▲ ABC and ADE, △ ADC is common.

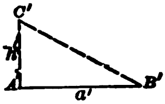
$$\triangle DBC \Leftrightarrow \triangle DCE. \qquad (§\ 369)$$

$$\therefore \triangle ABC \Leftrightarrow \triangle ADE. \qquad (\text{Ax. } 2)$$

Ex. 354. To transform a given triangle into an equivalent right triangle, having one leg equal to a given length.

Let ABC be the given △, a its base, and a' the given length. It is required to transform △ ABC into an equivalent rt. △ having one leg equal to a'.





CONSTRUCTION. Draw h, the altitude of △ ABC. Find h', a fourth proportional to a', a, and h (§ 352). Take $A'B' = a'$. At A' erect a ⊥ $A'C' = h'$ (§ 271). Join $B'C'$.

△ $A'B'C'$ is the rt. △ required.

PROOF. $a' : a = h : h'$; and hence $a' \times h' = a \times h$.

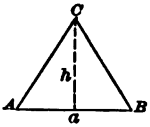
But $a' \times h' =$ twice the area of △ $A'B'C'$,

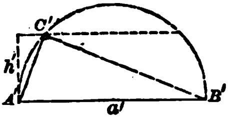
and $a \times h =$ twice the area of △ ABC. (§ 368)

$$\therefore \triangle A'B'C' \Leftrightarrow \triangle ABC.$$

Ex. 355. To transform a given triangle into an equivalent right triangle, having the hypotenuse equal to a given length.

Let ABC be the given △, a its base, and a' the given length. It is required to transform △ ABC into an equivalent rt. △, having its hypotenuse equal to a'.





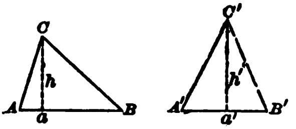
CONSTRUCTION. Draw h, the altitude of △ ABC. Find h', a fourth proportional to a', a, and h

(§ 352). Construct a rt. $\triangle A'B'C'$, with the hypotenuse, a', and the altitude upon the hypotenuse $= h'$ (Ex. 150). $\triangle A'B'C'$ is the $\triangle$ required.

PROOF. The proof is the same as in Ex. 354.

Ex. 356. To transform a given triangle into an equivalent isosceles triangle, having the base equal to a given length.

Let ABC be the given $\triangle$, a its base, and a' the given length. It is required to transform $\triangle ABC$ into an equivalent isosceles $\triangle$ having a base equal to a'.



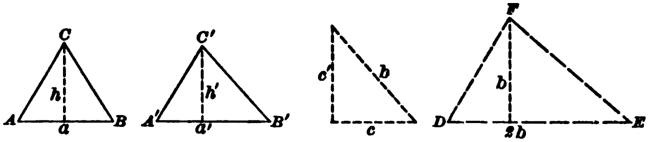
CONSTRUCTION. Draw h, the altitude of $\triangle ABC$. Find h', a fourth proportional to a', a, and h (§ 352). Take $A'B' = a'$. At the middle point of $A'B'$ erect a $\perp$ equal to h'. Join $A'C'$ and $B'C'$.

$\triangle A'B'C'$ is the $\triangle$ required.

PROOF. $A'C' = B'C'$ (§ 116); and hence $\triangle A'B'C'$ is isosceles.

The remainder of the proof is the same as in Ex. 354.

Ex. 357. To construct a triangle equivalent to the sum of two given triangles.



Let ABC and $A'B'C'$ be the given $\triangle$, and a and a' their bases. It is required to construct a $\triangle$ equivalent to the sum of $\triangle ABC$ and $A'B'C'$.

CONSTRUCTION. Draw h and h', the altitudes of $\triangle ABC$ and $A'B'C'$. Find c, a mean proportional between a and $\frac{1}{2}h$, and c', a mean proportional between a' and $\frac{1}{2}h'$ (§ 354). Find b so that $b^2 = c^2 + c'^2$ (§ 381). Construct $\triangle DEF$ with a base equal to $2b$, and an altitude equal to b.

$\triangle DEF$ is the $\triangle$ required.

PROOF. The area of $\triangle DEF = \frac{1}{2}(2b \times b)$ (§ 368)

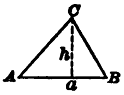
$= b^2 = c^2 + c'^2 = \frac{1}{2}(a \times h) + \frac{1}{2}(a' \times h')$. (Cons.)

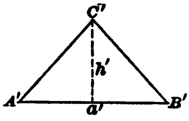
But the area of $\triangle ABC = \frac{1}{2}(a \times h)$,

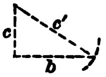
and the area of $\triangle A'B'C' = \frac{1}{2}(a' \times h')$. (§ 368)

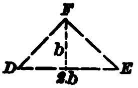
$\therefore \triangle DEF \approx \triangle ABC + \triangle A'B'C'$.

Ex. 358. To construct a triangle equivalent to the difference of two given triangles.









Let ABC and $A'B'C'$ be the given $\triangle$, and a and a' their bases. It is required to construct a $\triangle$ equivalent to the difference of $\triangle$ $A'B'C'$ and ABC.

Construction. Draw h and h', the altitudes of $\triangle$ ABC and $A'B'C'$; find c, a mean proportional between a and $\frac{1}{2}h$, and c', a mean proportional between a' and $\frac{1}{2}h'$ (§ 354). Find b so that $b^2 = c'^2 - c^2$ (§ 382). Construct $\triangle\ DEF$ with a base equal to $2b$, and an altitude equal to b.

$\triangle\ DEF$ is the $\triangle$ required.

Proof. The area of $\triangle\ DEF = \frac{1}{2}(2b \times b)$ (§ 368)

$= b^2 = c'^2 - c^2 = \frac{1}{2}(a' \times h') - \frac{1}{2}(a \times h)$. (Cons.)

But the area of $\triangle\ A'B'C' = \frac{1}{2}(a' \times h')$,

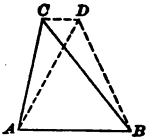
and the area of $\triangle\ ABC = \frac{1}{2}(a \times h)$. (§ 368)

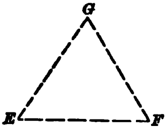
$$\therefore \triangle\ DEF \Bumpeq \triangle\ A'B'C' - \triangle\ ABC.$$

Ex. 359. To transform a given triangle into an equivalent equilateral triangle.

Let ABC be the given $\triangle$. It is required to transform $\triangle\ ABC$ into an equivalent equilateral $\triangle$.

Construction. Draw AD, making the $\angle\ BAD$ equal to $\frac{2}{3}$ of a rt. $\angle$ (Ex. 93), and meeting CD, drawn $\parallel$ to AB, at D. Join BD. Find EF, a mean proportional between AD and AB (§ 354). On EF construct an equilateral $\triangle\ EFG$ (§ 282). $\triangle\ EFG$ is the $\triangle$ required.





Proof. In the $\triangle$ ABD and EFG,

$\angle\ BAD = \angle\ E$. (§ 144)

$\triangle\ ABD : \triangle\ EFG = AD \times AB : EG \times EF$. (§ 374)

But $EG \times EF = \overline{EF}^2 = AD \times AB$. (Cons.)

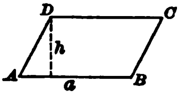
$$\therefore \triangle\ ABD \Bumpeq \triangle\ EFG.$$

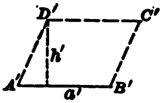
Also $\triangle\ ABD \Bumpeq \triangle\ ABC$. (§ 369)

$$\therefore \triangle\ EFG \Bumpeq \triangle\ ABC.$$

Ex. 360. To transform a parallelogram into a parallelogram having one side equal to a given length.

Let $ABCD$ be the given ▱, a its base, and a' the given length. It is required to transform ▱ $ABCD$ into an equivalent ▱ having one side equal to a'.





Construction. Draw h, the altitude of ▱ $ABCD$. Find h', a fourth proportional to a', a, and h (§ 352). Construct the ▱ $A'B'C'D'$ on a' as a base, with h' as an altitude. ▱ $A'B'C'D'$ is the ▱ required.

Proof. $a' : a = h : h'. \quad \therefore a' \times h' = a \times h.$

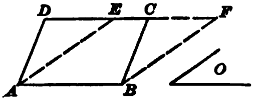
But the area of ▱ $A'B'C'D' = a' \times h'$,

and the area of ▱ $ABCD \quad = a \times h.$ (§ 365)

$\therefore$ ▱ $A'B'C'D' \Leftrightarrow$ ▱ $ABCD$.

Ex. 361. To transform a parallelogram into a parallelogram having one angle equal to a given angle.

Let $ABCD$ be the given ▱, and O the given ∠. It is required to transform ▱ $ABCD$ into an equivalent ▱ having an ∠ equal to ∠ O.

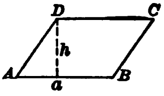


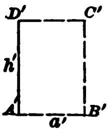
Construction. Construct the ∠ BAE equal to ∠ O. Draw $BF \parallel$ to AE, meeting DC produced at F. ▱ $ABFE$ is the ▱ required.

Proof. ▱ $ABFE \Leftrightarrow$ ▱ $ABCD$. (§ 366)

Ex. 362. To transform a parallelogram into a rectangle having a given altitude.

Let $ABCD$ be the given ▱, a its base, and h' the given altitude. It is required to transform the ▱ $ABCD$ into an equivalent rectangle having the altitude h'.





Construction. Draw h, the altitude of ▱ $ABCD$. Find a', a fourth proportional to h', h, and a (§ 352). Construct the rectangle $A'B'C'D'$, with the base a' and the altitude h'.

Rectangle $A'B'C'D'$ is the rectangle required.

Proof. $h' : h = a : a'. \quad \therefore a' \times h' = a \times h.$

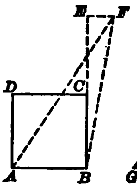
But $a' \times h' =$ the area of the rectangle $A'B'C'D'$, (§ 363)

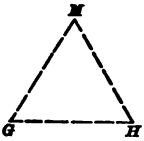
and $a \times h =$ the area of the ▱ $ABCD$. (§ 365)

$\therefore$ the rectangle $A'B'C'D' \Leftrightarrow$ ▱ $ABCD$.

Ex. 363. To transform a square into an equilateral triangle.

Let $ABCD$ be the given square. It is required to transform the square $ABCD$ into an equivalent equilateral $\triangle$.





Construction. Prolong BC to E, making $CE = BC$. Draw $EF \parallel$ to AB, and AF, making the $\angle BAF$ equal to two-thirds of a rt. $\angle$ (Ex. 93). Join BF. Construct the equilateral $\triangle GHM \Leftrightarrow \triangle ABF$ (Ex. 359). $\triangle GHM$ is the $\triangle$ required.

Proof. The area of $\triangle ABF = \frac{1}{2}(AB \times BE)$, (§ 368)

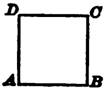
$$= \tfrac{1}{2}(AB \times 2BC) = \overline{AB}^2.$$

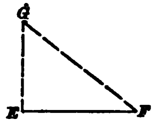
$$\therefore \triangle ABF \Leftrightarrow \text{the square } ABCD.$$

$$\therefore \triangle GHM \Leftrightarrow \text{the square } ABCD.$$

Ex. 364. To transform a square into a right triangle having one leg equal to a given length.

Let $ABCD$ be the given square, and EF the given length. It is required to transform the square $ABCD$ into an equivalent rt. $\triangle$ having one leg equal to EF.





Construction. Find EG, a fourth proportional to EF, $2AB$, and AB (§ 352). Construct the rt. $\triangle EFG$ with the legs EF and EG. $\triangle EFG$ is the $\triangle$ required.

Proof. $EF : 2AB = AB : EG.$

$$\therefore EF \times EG = 2\overline{AB}^2,$$

and $\frac{1}{2}(EF \times EG) = \overline{AB}^2.$

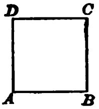
But $\frac{1}{2}(EF \times EG) =$ the area of $\triangle EFG$, (§ 368)

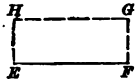
and $\overline{AB}^2 =$ the area of the square $ABCD$.

$$\therefore \triangle EFG \Leftrightarrow \text{the square } ABCD.$$

Ex. 365. To transform a square into a rectangle having one side equal to a given length.

Let $ABCD$ be the given square, and EF the given length. It is required to transform the square $ABCD$ into an equivalent rectangle having one side equal to EF.





Construction. Find EH, a third proportional to EF and AB (§ 353).

Construct the rectangle $EFGH$ with EF and EH as sides. Rectangle $EFGH$ is the rectangle required.

Proof. $EF : AB = AB : EH$.

$\therefore EF \times EH = \overline{AB}^2$.

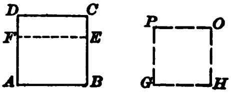
But $EF \times EH =$ the area of the rectangle $EFGH$, (§ 363)

and $\overline{AB}^2 =$ the area of the square $ABCD$.

$\therefore$ the rectangle $EFGH \Bumpeq$ the square $ABCD$.

Ex. 366. To construct a square equivalent to five-eighths of a given square.

Let $ABCD$ be the given square. It is required to construct a square equivalent to five-eighths of the square $ABCD$.



Construction. Divide BC into 8 equal parts (§ 279). Take $BE = \frac{5}{8} BC$. Draw EF ∥ to BA. Find GH, a mean proportional between AB and BE (§ 354). On GH construct the square $GHOP$.

Square $GHOP$ is the square required.

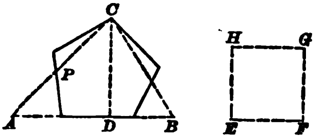
Proof. $AB : GH = GH : BE$. $\therefore AB \times BE = \overline{GH}^2$.

$\therefore$ the rectangle $ABEF \Bumpeq$ the square $GHOP$. (§ 363)

But the rectangle $ABEF \Bumpeq \frac{5}{8}$ of the square $ABCD$. (§ 367)

$\therefore$ the square $GHOP \Bumpeq \frac{5}{8}$ of the square $ABCD$.

Ex. 367. To construct a square equivalent to three-fifths of a given pentagon.



Let P represent the given pentagon. It is required to construct a square equivalent to three-fifths P.

Construction. Construct the $\triangle ABC \Bumpeq P$ (§ 386). Draw CD, the altitude of $\triangle ABC$. Divide AB into 5 equal parts (§ 279). Find EF, a mean proportional between $\frac{3}{5} AB$ and $\frac{1}{2} CD$ (§ 354). On EF construct the square $EFGH$.

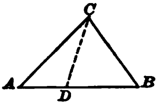
Square $EFGH$ is the square required.

Proof. $\frac{3}{5} AB : EF = EF : \frac{1}{2} CD$; and hence $\frac{1}{2}(\frac{3}{5} AB \times CD) = \overline{EF}^2$.

$\therefore$ the square $EFGH \Bumpeq \frac{3}{5} \triangle ABC$. (§ 368)

$\therefore$ the square $EFGH \Bumpeq \frac{3}{5} P$.

Ex. 368. To draw a line through the vertex of a given triangle so as to divide the triangle into two triangles which shall be to each other as 2 : 3.



Let ABC be the given $\triangle$. It is required to draw a line through the vertex C, so as to divide $\triangle\, ABC$ into two $\triangle$ which shall be to each other as 2 : 3.

Construction. Divide AB into 5 equal parts (§ 279). Let $AD = 2$ parts, and $DB = 3$ parts. Draw CD.

CD is the line required.

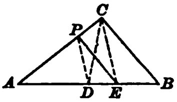
Proof. $\triangle\, ACD : \triangle\, BCD = AD : BD.$ (§ 370)

But $AD : DB = 2 : 3.$ (Cons.)

$\therefore \triangle\, ACD : \triangle\, BCD = 2 : 3.$

Ex. 369. To divide a given triangle into two equivalent parts by drawing a line through a given point P in one of the sides.

Let ABC be the given $\triangle$, and P the given point. It is required to draw a line through P so as to divide $\triangle\, ABC$ into two equivalent parts.



Construction. Draw CD to the middle point of AB. Join PD. Draw $CE \parallel PD$. Join PE.

PE is the line required.

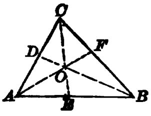
Proof. $\triangle\, ADC \Leftrightarrow \triangle\, BDC$ (§ 369); and hence $\triangle\, ADC \Leftrightarrow \frac{1}{2} \triangle\, ABC.$

In the $\triangle$ ADC and AEP, $\triangle\, ADP$ is common,

and $\triangle\, DEP \Leftrightarrow \triangle\, DPC.$ (§ 369)

$\therefore \triangle\, AEP \Leftrightarrow \triangle\, ADC \Leftrightarrow \frac{1}{2} \triangle\, ABC.$

Ex. 370. To find a point within a triangle, such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.



Let ABC be the given $\triangle$. It is required to find a point within $\triangle\, ABC$ such that the line joining this point to the vertices shall divide the $\triangle$ into three equivalent parts.

Construction. Draw the medians AF, BD, and CE; they will meet at the point O (Ex. 33).

O is the point required.

Proof. $\triangle\, AEC \Leftrightarrow \frac{1}{2} \triangle\, ABC$, and $\triangle\, ABD \Leftrightarrow \frac{1}{2} \triangle\, ABC.$ (§ 370)

$\therefore \triangle\, AEC \Leftrightarrow \triangle\, ABD.$

Subtract from each $\triangle$ the common quadrilateral $AEOD$; then

$$\triangle EBO \Bumpeq \triangle DOC. \qquad (1)$$

$$\triangle AEO \Bumpeq \triangle EBO, \text{ and } \triangle AOD \Bumpeq \triangle DOC. \qquad (§\ 369)$$

$$\therefore \triangle AEO \Bumpeq \triangle AOD. \qquad (2)$$

Add (1) and (2); then

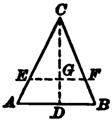
$$\triangle AEO + \triangle EBO \Bumpeq \triangle AOD + \triangle DOC.$$

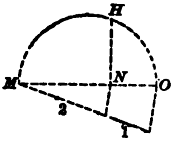
Or $$\triangle ABO \Bumpeq \triangle AOC.$$

In the same way it may be proved that $\triangle ABO \Bumpeq \triangle BOC$.

Ex. 371. To divide a given triangle into two equivalent parts by drawing a line parallel to one of the sides.

Let ABC be the given $\triangle$. It is required to divide $\triangle ABC$ into two equivalent parts by drawing a line $\parallel$ to AB.





Construction. Draw CD, the altitude of $\triangle ABC$. Take $MN = CD$. Find NH, so that

$$\overline{MN}^2 : \overline{NH}^2 = 2 : 1. \ (§\ 387)$$

Take $CG = NH$. Draw EF through $G \parallel$ to AB. EF is the line required.

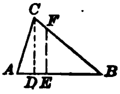
Proof. Since EF is $\parallel$ to AB, $\triangle$s ABC and EFC are mutually equiangular and similar (§ 321).

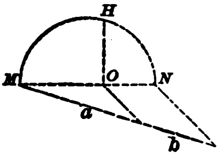
$$\therefore \triangle ABC : \triangle EFC = \overline{CD}^2 : \overline{CG}^2 \qquad (§\ 377)$$

$$= \overline{MN}^2 : \overline{NH}^2 = 2 : 1.$$

Ex. 372. To divide a given triangle into two equivalent parts by drawing a line perpendicular to one of the sides.

Let ABC be the given $\triangle$. It is required to divide $\triangle ABC$ into two equivalent parts by drawing a line $\perp$ to one of the sides.





Construction. Draw CD, the altitude of $\triangle ABC$. Take $a = 2DB$, $b = AB$, $MO = BC$.

Find OH, so that

$$\overline{MO}^2 : \overline{OH}^2 = a : b. \qquad (§\ 387)$$

Cut off $BF = OH$, and draw $FE \perp$ to AB. FE is the line required.

Proof. $$\frac{\triangle ABC}{\triangle DBC}=\frac{AB}{DB}. \quad (1) \qquad (\S\ 370)$$

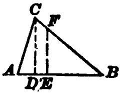
Since FE and CD are $\perp$ to AB, they are ∥.

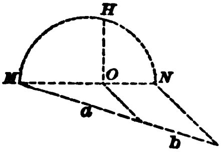
∴ ▲ DBC and EBF are similar. (§ 321)

$$\therefore \frac{\triangle DBC}{\triangle EBF}=\frac{\overline{BC}^2}{\overline{BF}^2} \qquad (\S\ 375)$$

$$=\frac{\overline{MO}^2}{\overline{OH}^2}=\frac{a}{b}$$

$$=\frac{2DB}{AB}. \qquad \text{(Cons.)}$$



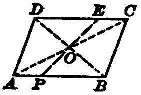


$$\frac{\triangle DBC}{\triangle EBF}=\frac{2DB}{AB}. \quad (2)$$

Multiply (1) and (2), $$\frac{\triangle ABC}{\triangle EBF}=\frac{2}{1}.$$

Ex. 373. To divide a given parallelogram into two equal parts by drawing a line through a given point in one of the sides.

Let $ABCD$ be the given ▱, and P the given point. It is required to divide ▱ $ABCD$ into two equal parts by drawing a line through the point P.

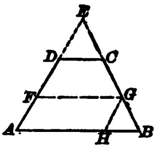


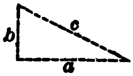
Construction. Draw the diagonals AC and BD, intersecting at O. Through O draw PE.

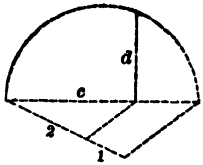
PE is the line required.

Proof. The quadrilateral $APED$ = the quadrilateral $PBCE$. (Ex. 316)

Ex. 374. To divide a given trapezoid into two equivalent parts by drawing a line parallel to the bases.







Let $ABCD$ be the given trapezoid. It is required to divide the trapezoid $ABCD$ into two equivalent parts by drawing a line ∥ to AB.

CONSTRUCTION. Draw a rt. $\triangle$ with the side $a = AB$, and the side $b = DC$. Find d so that $d^2 : c^2 = 1 : 2$ (§ 387). On AB take $AH = d$. Draw $HG \parallel$ to AD, and $FG \parallel$ to AB.

FG is the line required.

PROOF. Produce AD and BC until they meet at E.

$$FG = AH \text{ (§ 180)} = d.$$

Since DC, FG, and AB are $\parallel$, $\triangle$s ABE, FGE, and DCE are similar.

$$\therefore \triangle FGE : \triangle DCE = \overline{FG}^2 : \overline{DC}^2, \quad (1)$$

and $$\triangle ABE : \triangle DCE = \overline{AB}^2 : \overline{DC}^2. \quad (2) \qquad \text{(§ 375)}$$

From (1), by division,

$$\triangle FGE - \triangle DCE : \triangle DCE = \overline{FG}^2 - \overline{DC}^2 : \overline{DC}^2.$$

But $$\overline{FG}^2 = \tfrac{1}{2}c^2 = \tfrac{1}{2}(a^2 + b^2) = \tfrac{1}{2}(\overline{AB}^2 + \overline{DC}^2). \qquad \text{(Cons.)}$$

Substituting for $\overline{FG}^2$, $\frac{1}{2}(\overline{AB}^2 + \overline{DC}^2)$,

$$\triangle FGE - \triangle DCE : \triangle DCE = \tfrac{1}{2}(\overline{AB}^2 - \overline{DC}^2) : \overline{DC}^2. \quad (3)$$

From (2), by division,

$$\triangle ABE - \triangle DCE : \triangle DCE = \overline{AB}^2 - \overline{DC}^2 : \overline{DC}^2. \quad (4)$$

From (3) and (4),

$$\triangle FGE - \triangle DCE : \triangle ABE - \triangle DCE = \tfrac{1}{2}(\overline{AB}^2 - \overline{DC}^2) : \overline{AB}^2 - \overline{DC}^2. \quad (5)$$

But $$\triangle FGE - \triangle DCE = \text{the trapezoid } FGCD,$$

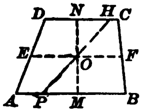
and $$\triangle ABE - \triangle DCE = \text{the trapezoid } ABCD.$$

Therefore in (5), $FGCD : ABCD = \frac{1}{2} : 1$.

$$\therefore \text{the trapezoid } FGCD \Leftrightarrow \tfrac{1}{2} \text{ the trapezoid } ABCD.$$

Ex. 375. To divide a given trapezoid into two equivalent parts by drawing a line through a given point in one of the bases.

I. Let $ABCD$ be the given trapezoid and P the given point, and suppose $$AP > AB - DC.$$



It is required to divide the trapezoid $ABCD$ into two equivalent parts by drawing a line through P.

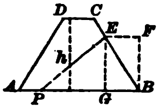
CONSTRUCTION. Draw the median EF and the altitude NM. Through O, the middle point of EF, draw PH.

PH is the line required.

PROOF. The trapezoids $APHD$ and $PBCH$ have the equal medians EO and OF, and the same altitude NM.

$$\therefore \text{the trapezoid } APHD \Leftrightarrow \text{the trapezoid } PBCH. \qquad \text{(§ 372)}$$

II. Let $ABCD$ be the given trapezoid, and P the given point, and suppose $AP < AB - DC$.



Construction. Draw the altitude h. Find BF, a fourth proportional to $\frac{1}{2}BP$, $\frac{1}{2}(AB + DC)$, and h. Draw $BF \perp$ to AB at B, $FE \parallel$ to BA, and $EG \parallel$ to FB. Join PE. PE is the line required.

Proof. $EG = BF$ (§ 180); and hence $\frac{1}{2}BP : \frac{1}{2}(AB + DC) = h : EG$.

$$\therefore \tfrac{1}{2}(BP \times EG) = \tfrac{1}{2}(AB + DC)h.$$

The area of $\triangle PBE = \frac{1}{2}(BP \times EG)$. (§ 368)

The area of the trapezoid $ABCD = \frac{1}{2}(AB + DC)h$. (§ 371)

$$\therefore \triangle PBE \Bumpeq \tfrac{1}{2} \text{ the trapezoid } ABCD.$$

BOOK V.

Ex. 376. The area of a circumscribed square is equal to twice the area of the inscribed square.

PROOF. Let S be the side of the circumscribed square, s the side and d the diagonal of the inscribed square.

Then $$\frac{d}{s} = \sqrt{2}; \qquad (\S\ 340)$$

but $$d = \text{diameter of circle} = S.$$

$$\therefore \frac{S}{s} = \sqrt{2}, \text{ or } \frac{S^2}{s^2} = 2.$$

$$\therefore S^2 = 2s^2.$$

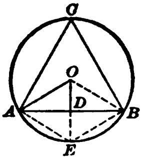
Ex. 377. If the length of the side of an inscribed square is 2 inches, what is the length of the side of the circumscribed square?

SOLUTION. Let S be the side of the circumscribed square, and s the side of the inscribed square.

Then $$\frac{S}{s} = \sqrt{2}. \qquad (\text{Ex. } 376)$$

$$\therefore \frac{S}{2} = \sqrt{2}, \text{ or } S = 2\sqrt{2}.$$

Ex. 378. In a regular inscribed triangle $a = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.



Let ABC be an equilateral triangle inscribed in the circle whose centre is O.

Let OD be perpendicular to AB.

Let $AO = R$, $OD = r$, $AB = a$, $\angle CAB = A$, and $\angle AOB = C$.

I. To prove $$a = R\sqrt{3}.$$

PROOF. Produce OD to meet the arc AB in E. Join AE, BO, BE. E is the middle of the arc AB. (§ 232)

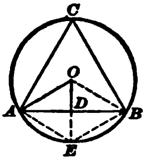
$$AO = AE = BO = BE; \qquad (\S\ 431)$$

hence AB and OE mutually bisect each other in D. (§ 123)

$$\overline{AD}^2 = \overline{AO}^2 - \overline{OD}^2. \qquad (\S\ 339)$$

That is, $$(\tfrac{1}{2}a)^2 = R^2 - (\tfrac{1}{2}R)^2.$$

$$\therefore a = R\sqrt{3}.$$



II. To prove $r = \frac{1}{2}R.$

Proof. Since AB bisects OE,

$$\therefore\ OD = \tfrac{1}{2}OE, \text{ or } r = \tfrac{1}{2}R.$$

III. To prove $A = 60°.$

Proof. Since ABC is an equilateral △, it is equiangular. (§ 155)

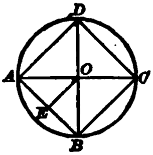
$$\therefore\ A = \tfrac{1}{3} \text{ of } 180° = 60°.$$

IV. To prove $C = 120°.$

Proof. Since $\angle AOB$ is measured by $\frac{1}{3}$ of a circumference, it is $\frac{1}{3}$ of $360° = 120°.$

Ex. 379. In an inscribed square,

$$a = R\sqrt{2},\ r = \tfrac{1}{2}R\sqrt{2},\ A = 90°,\ C = 90°.$$



Let $ABCD$ be an inscribed square, O the centre, and OE the apothem.

I. To prove $a = R\sqrt{2}.$

Proof. $AB = AO\sqrt{2};$ (§ 340)

that is, $a = R\sqrt{2}.$

II. To prove $r = \frac{1}{2}R\sqrt{2}.$

Proof. Since $\angle OAE = 45°$, $AE = OE.$

Hence
$$OE = \frac{AO}{\sqrt{2}} = \frac{AO}{2}\sqrt{2};$$

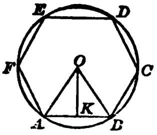
that is, $r = \frac{1}{2}R\sqrt{2}.$

III. $A = 90°$ because it is measured by $\frac{1}{2}$ a semi-circumference.

IV. $C = 90°$ because it is measured by $\frac{1}{4}$ a circumference.

Ex. 380. In a regular inscribed hexagon

$$a = R,\ r = \tfrac{1}{2}R\sqrt{3},\ A = 120°, \text{ and } C = 60°.$$



Let $ABCDEF$ be a regular hexagon inscribed in a ⊙ whose centre is O. Let OK be the apothem. Then $AO = R$, $AB = a$, and $OK = r.$

I. $a = R.$ (§ 431)

II. To prove $r = \frac{1}{2}R\sqrt{3}.$

Proof. $\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2;$ (§ 339) that is, $$r^2 = R^2 - (\tfrac{1}{2}R)^2 = \frac{3R^2}{4}.$$

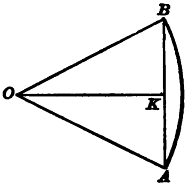
$$\therefore r = \tfrac{1}{2}R\sqrt{3}.$$

III. $A = \angle FAB = 120°$, since it is measured by $\frac{1}{2}$ of $\frac{4}{6}$ of the circumference.

IV. $C = \angle AOB = 60°$, since it is measured by $\frac{1}{6}$ of the circumference.

Ex. 381. In a regular inscribed decagon

$$a = \frac{R(\sqrt{5}-1)}{2},\quad r = \tfrac{1}{4}R\sqrt{10+2\sqrt{5}},\quad A = 144°,\ \text{and}\ C = 36°.$$



Let AB be one side of a regular decagon inscribed in a ⊙ whose centre is O. Let OK be the apothem.

Then $OA = R$, $OK = r$, and $AB = a$.

I. To prove $$a = \frac{R(\sqrt{5}-1)}{2}.$$

Proof. $OA : AB :: AB : OA - AB.$ (§ 434)

That is, $$R : a :: a : R - a.$$

$$\therefore a = \frac{R(\sqrt{5}-1)}{2}.$$

II. To prove $$r = \tfrac{1}{4}R\sqrt{10+2\sqrt{5}}.$$

Proof. $$\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2.$$ (§ 339)

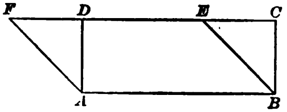
That is, $$r^2 = R^2 - (\tfrac{1}{2}a)^2 = R^2 - \left\{\frac{R(\sqrt{5}-1)}{4}\right\}^2$$

$$\therefore r = \tfrac{1}{4}R\sqrt{10+2\sqrt{5}}.$$

III. $A = 2\angle OAB = 144°$, since it is measured by $\frac{1}{2}$ of $\frac{8}{10}$ of the circumference.

IV. $C = \angle AOB = 36°$, since it is measured by $\frac{1}{10}$ of the circumference.

Ex. 382. Of all equivalent parallelograms having equal bases, the rectangle has the least perimeter.



Let $ABCD$ be a rectangle, and $ABEF$ any other ▱ on the same base AB, and having the same altitude.

To prove $$2AB + 2BC < 2AB + 2BE.$$

Proof. $BC < BE$ (§ 114); and hence $2BC < 2BE.$

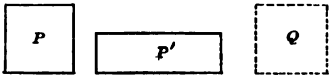
Adding $2AB$ to each side of this inequality, we have

$$2AB + 2BC < 2AB + 2BE.$$

Ex. 383. Of all rectangles of a given area, the square has the least perimeter.

Let P be a square, and P' any other rectangle equivalent to P.

To prove perimeter of P less than perimeter of P'.



Proof. Construct the square Q, having the same perimeter as P'.

Then $Q > P'$. (§ 450)

But $P \approx P'$. $\therefore Q > P$.

Since Q is greater than P, and they are both squares, the perimeter of Q must be greater than the perimeter of P.

But the perimeter of P' equals the perimeter of Q, by construction; therefore the perimeter of P' is greater than the perimeter of P.

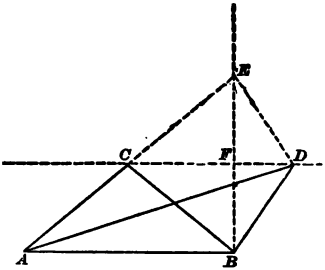
Ex. 384. Of all triangles upon the same base, and having the same altitude, the isosceles triangle has the least perimeter.

Let ABC be an isosceles $\triangle$, and ABD any other $\triangle$, having AB for a base, and its vertex D in a line CD $\parallel$ to AB.

To prove

$AB + AC + CB < AB + AD + DB$.

Proof. Draw $BE \perp$ to AB, meeting AC produced in E. Join DE.



BE is $\perp$ to CD. (§ 102)

$\angle ECF = \angle CAB$ (§ 106)

$= \angle CBA$ (§ 154)

$= \angle BCF$. (§ 104)

$\therefore$ the rt. $\triangle$ $ECF =$ rt. $\triangle$ BCF. (§ 149)

$\therefore FE = FB$.

$\therefore CE = CB$ and $DE = DB$. (§ 122)

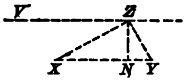
But $AE < AD + DE$, or $AC + CB < AD + DB$. (§ 137)

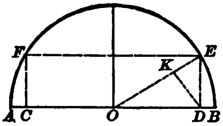
$\therefore AB + AC + CB < AB + AD + DB$.

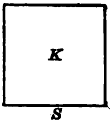
Ex. 385. To divide a straight line into two parts such that their product shall be a maximum.

Proof. From § 392, it is evident that the rectangle $CM \times CN$ will be a maximum when MP is equal to the radius; in which case $CM = CN$.

Ex. 386. To inscribe in a semicircle a rectangle having a given area; a rectangle having the maximum area.







Let $AFEB$ be a semicircle, and K a given square whose side is S.

To inscribe in $AFEB$ a rectangle having an area equal to K; also to inscribe a maximum rectangle in $AFEB$.

Construction. Find p, a third proportional to AB and S. Take $XY = OB$. Draw $VZ \parallel$ to XY, and at the distance p from XY. Upon XY as a diameter, describe a semicircle cutting VZ at Z. Take $OD = XZ$, draw $DE \perp$ to AB, cutting the semi-circumference in E, and complete the rectangle CE. CE will be the rectangle required.

Proof. Draw $DK \perp$ to OE.

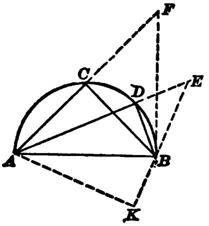
Rectangle $CE = 4 \times \triangle ODE = 2\,OE \times DK = AB \times p = S^2 = K$.

In order that rectangle CE may be a maximum, $\triangle ODE (= \triangle XYZ)$ must be a maximum; in which case $DK (= NZ)$ is a maximum.

But $NZ (= DK)$ has its greatest length when $NX = NY$ (when $KO = KE$); that is, when $\angle ZXY (= \angle EOD) = 45°$.

Hence to inscribe the maximum rectangle in a semicircle, draw a radius OE, making an $\angle$ of 45° with OB; then rectangle CE will be the required maximum rectangle.

Ex. 387. To find a point in a semi-circumference such that the sum of its distances from the extremities of the diameter shall be a maximum.



Let ACB be a semicircle, C the middle point of the arc ACB, and D any other point.

To prove $AC + BC > AD + BD$.

Proof. Produce AC to F, making $CF = AC = CB$. Join BF. Produce AD to E, making $DE = DB$. Draw EB, and produce it to meet AK drawn through $A \perp$ to EB.

Since $CA = CB = CF$, B is on the semi-circumference on AF as a diameter.

$$\therefore ABF \text{ is a rt. } \angle.$$

$$\angle ACB = \angle F + \angle CBF = 2\angle F. \quad \therefore \angle F = \tfrac{1}{2} ACB = \tfrac{1}{2} ADB. \qquad (\S\ 267)$$

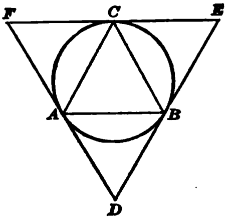
$$\angle ADB = \angle E + \angle DBE = 2\angle E. \quad \therefore \angle E = \tfrac{1}{2} ADB = \angle F.$$

$$\therefore \text{The } \triangle\text{s } AFB \text{ and } AEK \text{ are similar.} \qquad (\S\ 323)$$

But $$AK < AB. \qquad (\S\ 114)$$

$$\therefore AE < AF, \text{ or } AD + BD < AC + BC.$$

Ex. 388. The side of a circumscribed equilateral triangle is equal to twice the side of the similar inscribed triangle. Find the ratio of their areas.



Let DEF be a circumscribed equilateral $\triangle$, and ABC an inscribed equilateral $\triangle$.

To prove $FE = 2\,CB$, and to find the ratio of the areas DEF and ABC.

Proof. Since EB and EC are tangents from E, they are equal. (§ 246)

$$\therefore \angle ECB = \angle EBC.$$

But $$\angle E = 60°.$$

$$\therefore \angle EBC = \angle ECB = 60° = \angle E.$$

$$\therefore EC = CB.$$

$$\therefore EF = 2\,CB.$$

$$\frac{\text{area } DEF}{\text{area } ABC} = \frac{\overline{FE}^2}{\overline{CB}^2} = 2^2 = 4.$$

Ex. 389. The apothem of an inscribed equilateral triangle is equal to half the radius of the circle.

Proof. The side of the inscribed regular hexagon equals the radius (§ 431).

The apothem of the inscribed equilateral $\triangle = \frac{1}{2}$ the radius (Ex. 378). Hence the theorem.

Ex. 390. The apothem of an inscribed regular hexagon is equal to half the side of the inscribed equilateral triangle.

Proof. The apothem of the hexagon $= \frac{1}{2}R\sqrt{3}$. (Ex. 380)

The side of the $\triangle = R\sqrt{3}$. (Ex. 378)

Hence the theorem.

Ex. 391. The area of an inscribed regular hexagon is equal to three-fourths of that of the circumscribed regular hexagon.

Proof. The apothem of the circumscribed hexagon is the radius $= R$. The apothem of the inscribed hexagon $= \frac{1}{2}R\sqrt{3}$ (Ex. 380).

$$\therefore \frac{\text{area inscribed hexagon}}{\text{area circumscribed hexagon}} = \frac{(\frac{1}{2}R\sqrt{3})^2}{R^2} = \frac{3R^2}{4R^2} = \frac{3}{4}.$$

Ex. 392. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Proof. Area of inscribed equilateral $\triangle$ whose side is a

$$=\frac{a}{2}\times\tfrac{3}{2}R=\frac{R\sqrt{3}}{2}\times\frac{3R}{2}=\tfrac{3}{4}R^2\sqrt{3}. \qquad \text{(Ex. 378)}$$

Area of circumscribed equilateral $\triangle$

$$=4\times\tfrac{3}{4}R^2\sqrt{3}=3R^2\sqrt{3}. \qquad \text{(Ex. 388)}$$

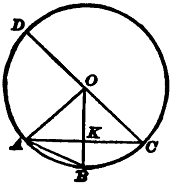
The mean proportional between these areas

$$=\sqrt{\tfrac{3}{4}R^2\sqrt{3}\times 3R^2\sqrt{3}}=\tfrac{3}{2}R^2\sqrt{3}.$$

The area of a regular inscribed hexagon

$$=\frac{6R}{2}\times\tfrac{1}{2}R\sqrt{3}=\tfrac{3}{2}R^2\sqrt{3}. \qquad \text{(Ex. 380)}$$

Ex. 393. The area of an inscribed regular octagon is equal to that of a rectangle whose sides are equal to the sides of the inscribed and the circumscribed squares.



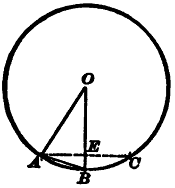
Let AB be the side of an inscribed octagon whose area is P, AC the side of the inscribed square, and CD the diameter of the circle or the side of the circumscribed square.

To prove $P=CD\times AC$.

Proof. Area of $\triangle ABO=\frac{1}{2}BO\times AK=\frac{1}{8}CD\times AC$ (since $BO=\frac{1}{2}CD$ and $AK=\frac{1}{2}AC$).

But the $\triangle\ ABO$ is $\frac{1}{8}$ of the octagon; hence $P=8\times\frac{1}{8}CD\times AC=CD\times AC$.

Ex. 394. The area of an inscribed regular dodecagon is equal to three times the square of the radius.



Let AB be the side of a regular dodecagon whose area is P inscribed in a circle whose centre is O and radius R.

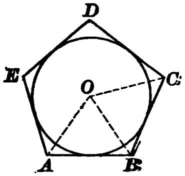
To prove $P=3R^2$.

Proof. Draw AC, a side of a regular inscribed hexagon.

Area of $\triangle AOB=OB\times\frac{1}{2}AE=R\times\frac{1}{4}R=\frac{1}{4}R^2$.

But the $\triangle\ AOB$ is $\frac{1}{12}$ of the dodecagon; hence $P=12\times\frac{1}{4}R^2=3R^2$.

Ex. 395. Every equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.



Let $ABCDE$ be an equilateral circumscribed polygon of an odd number of sides.

To prove $ABCDE$ regular.

Proof. Join the centre O with A, B, C.

Then OA, OB, OC bisect the angles A, B, C, respectively (§ 246).

$$\triangle OBA = \triangle OBC. \qquad (§ 150)$$

$$\therefore \angle OAB = \angle OCB.$$

$$\therefore \angle BAE = \angle BCD. \qquad (\text{Ax. } 6)$$

In like manner it may be proved that the first of every three consecutive ∠s is equal to the third.

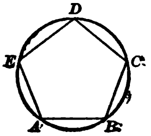
That is, $\angle A = \angle C = \angle E = \angle B = \angle D.$

Hence the polygon is regular, since it is equilateral and equiangular.

The same method is manifestly applicable in every case where the number of sides is odd.

Ex. 396. Every equiangular polygon inscribed in a circle is regular if it has an odd number of sides.

Let $ABCDE$ be an inscribed equiangular polygon having an odd number of sides.



To prove $ABCDE$ regular.

Proof. Since the equal inscribed ∠s intercept equal arcs on the circumference, if each of these arcs be subtracted from the circumference, the remainders will be equal.

That is, $\text{arc } ABC = \text{arc } BCD.$

Taking arc BC from each of these equals, we have

$$\text{arc } AB = \text{arc } CD.$$

In like manner

$$\text{arc } CD = \text{arc } EA = \text{arc } BC = \text{arc } DE = \text{arc } AB.$$

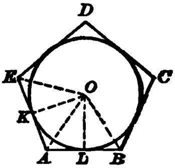
$$\therefore AB = BC = CD, \text{ etc.} \qquad (§ 230)$$

Hence the polygon is equiangular and equilateral, and therefore regular.

The same method is manifestly applicable to any polygon of an odd number of sides.

Ex. 397. Every equiangular polygon circumscribed about a circle is regular.

Let $ABCDE$ be any equiangular polygon circumscribed about the ⊙ whose centre is O.



To prove $ABCDE$ regular.

Proof. Join OA, OB. Draw OK and OL to the points of contact of the sides AE and AB respectively.

$$\text{rt. } \triangle OAK = \text{rt. } \triangle OAL. \qquad (\S\ 161)$$

$$\therefore AK = AL.$$

In the rt. ▲ OLA and OLB, $\angle OAL = \angle OBL$ (being halves of the equal ∠s A and B).

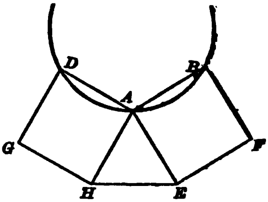
$$\therefore \triangle OLA = \triangle OLB. \quad \therefore LA = LB. \quad \therefore AB = 2AL.$$

In like manner $AE = 2AK$. But $AL = AK$. $\therefore AB = AE$.

In like manner $AB = BC = CD = DE = EA$.

Hence the polygon is regular.

Ex. 398. Upon the six sides of a regular hexagon squares are constructed outwardly. Prove that the exterior vertices of these squares are vertices of a regular dodecagon.



Let DA and AB be consecutive sides of a regular inscribed hexagon, and AG and AF squares on AD and AB.

To prove G, H, E, F, etc., vertices of a regular dodecagon.

Proof. Draw HE.

$$\angle DAB = 120^\circ. \qquad (\text{Ex. } 380)$$

$$\angle HAE = 360^\circ - (DAB + DAH + BAE) = 360^\circ - (120^\circ + 90^\circ + 90^\circ) = 60^\circ.$$

Also, $AH = AE$, since they are sides of equal squares.

$$\therefore \angle AHE = \angle AEH.$$

But $$\angle HAE = 60^\circ.$$

$$\therefore \angle AHE = \angle AEH = 60^\circ.$$

$$\therefore AE = EH.$$

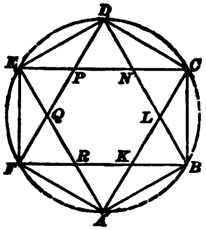
$$\therefore GH = HE = EF, \text{ etc.}$$

That is, the dodecagon is equilateral.

Now, $\angle GHE = \angle HEF$, etc., since each is composed of a right angle and an angle of 60°. That is, the dodecagon is equiangular.

Hence G, H, E, F, etc., are the vertices of a regular dodecagon.

Ex. 399. The alternate vertices of a regular hexagon are joined by straight lines. Prove that another regular hexagon is thereby formed. Find the ratio of the areas of the two hexagons.



Let ABC..... be a regular hexagon. Let BD, CE, etc., join the alternate vertices and intersect in N, P, Q, R, K, and L.

To prove $NPQRKL$ a regular hexagon.

Proof. Draw the circumscribed ⊙ $ABCDEF$.

$$\angle DFB = \angle AEC = 60°$$

(each being measured by ½ of ⅓ of the circumference).

$\angle EPQ$ is measured by $\frac{1}{2}$(arc EF + arc DC) (§ 268) $= 60°$.

$\angle EQP$ is measured by $\frac{1}{2}$(arc ED + arc FA) (§ 268) $= 60°$.

Hence $\triangle EPQ$ is equilateral. (§ 157)

In like manner FQR, ARK, etc., are equilateral.

$$\angle QEF = \angle QFE \text{ (since arc } FA = \text{arc } ED).$$

$$\therefore QE = QF.$$

But $QE = QP$, and $QF = QR$.

$$\therefore QP = QR.$$

In like manner $QR = RK = KL$, etc.

That is, $NPQRKL$ is equilateral.

Now, $\angle PQR$ is measured by $\frac{1}{2}$(arc EF + arc DBA) $= \frac{1}{2}$ of $\frac{2}{3}$ of the circumference.

$$\therefore \angle PQR = 120°.$$

In like manner each ∠ of $NPQRKL$ may be proven equal to 120°.

Hence $NPQRKL$ is regular.

Since it has been shown that $DP = PQ = QF$,

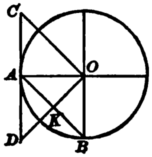
$$\therefore QP = \tfrac{1}{3} DF = \tfrac{1}{3} R\sqrt{3}. \qquad \text{(Ex. 378)}$$

But $DE = R$. (§ 431)

$$\therefore \frac{ABCDEF}{NPQRKL} = \frac{R^2}{(\frac{1}{3} R\sqrt{3})^2} = 3.$$

That is, the outer hexagon is 3 times the inner hexagon.

Ex. 400. The radius of an inscribed regular polygon is a mean proportional between the apothem and the radius of the similar circumscribed regular polygon.



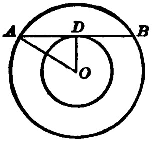
Let OA be the radius and OK the apothem of the inscribed regular polygon, of which AB is a side. Let OD be the radius of the similar circumscribed regular polygon of which CD is a side.

To prove $OD : OA :: OA : OK$.

Proof. Since CD is tangent to the $\odot$ at A, $\angle OAD$ is a rt. $\angle$. But AK is $\perp$ to OD.

$\therefore OD : OA :: OA : OK$. (§ 334)

Ex. 401. The area of a circular ring is equal to that of a circle whose diameter is a chord of the outer circle and a tangent to the inner circle.



Let AB be a tangent to the inner $\odot$ whose radius is $OD = r$, and let AB be a chord to the outer $\odot$ whose radius is $OA = R$.

To prove area of ring $= \frac{1}{4}\pi \times \overline{AB}^2$.

Proof. Area of outer circle $= \pi R^2$. (§ 425)

Area of inner circle $= \pi r^2$. (§ 425)

$\therefore$ Area of ring $= \pi(R^2 - r^2)$.

But $\overline{OA}^2 - \overline{OD}^2 = \overline{AD}^2 = \frac{1}{4}\overline{AB}^2$.

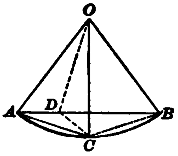
That is, $R^2 - r^2 = \frac{1}{4}\overline{AB}^2$.

$\therefore \pi(R^2 - r^2) = \frac{1}{4}\pi \times \overline{AB}^2$.

But $\pi(R^2 - r^2) =$ area of ring.

$\therefore \frac{1}{4}\pi \times \overline{AB}^2 =$ area of ring.

Ex. 402. The square on the side of an inscribed regular pentagon is equal to the sum of the squares of the radius of the circle and the side of the inscribed regular decagon.



Let AB be the side of a regular inscribed pentagon, and AC the side of a regular inscribed decagon.

To prove $\overline{AC}^2 + \overline{OB}^2 = \overline{AB}^2$.

Proof. Draw OD to bisect the $\angle AOC$ and meeting AB in D. Join DC and BC.

In the $\triangle$ OAD and OCD,

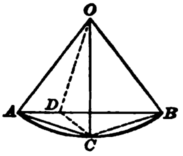
$OA = OC$, $OD = OD$, $\angle AOD = \angle COD$. (Cons.)

$\therefore \triangle OAD = \triangle OCD$. (§ 150)

$\therefore AD = DC$.

That is, $\triangle ADC$ is isosceles and similar to the isosceles $\triangle ACB$, since the $\angle CAB$ is common.

$\therefore AB : AC :: AC : AD$, or $\overline{AC}^2 = AB \times AD$. (1)



Again, $\angle AOB = \frac{1}{5}$ of $360° = 72°$.

$\angle AOC = \frac{1}{2}$ of $72° = 36°$.

$\angle COD = \frac{1}{2}$ of $36° = 18°$.

$\therefore \angle DOB = 36° + 18° = 54°$.

But $\angle OAB = \frac{1}{2}(180° - AOB)$

$= \frac{1}{2}(180° - 72°) = 54°$.

That is, $\triangle\ DOB$ is isosceles and similar to $\triangle\ OAB$, having $\angle\ OBD$ in common.

$$\therefore AB : OB :: OB : BD, \text{ or } \overline{OB}^2 = AB \times BD. \qquad (2)$$

Adding equations (1) and (2), we have

$$\overline{OB}^2 + \overline{AC}^2 = AB(BD + DA) = AB \times AB = \overline{AB}^2.$$

Ex. 403. In a regular pentagon, if a is a side, and R the radius of the circumscribed circle, to prove

$$a = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}.$$

Proof. The side of an inscribed decagon

$$= \frac{R(\sqrt{5} - 1)}{2}. \qquad \text{(Ex. 381)}$$

Hence
$$a^2 = R^2 + \left\{\frac{R(\sqrt{5} - 1)}{2}\right\}^2 \qquad \text{(Ex. 402)}$$

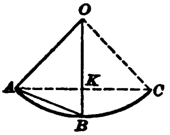
$$= \frac{4R^2 + R^2(6 - 2\sqrt{5})}{4}$$

$$= \frac{R^2(10 - 2\sqrt{5})}{4}.$$

$$\therefore a = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}.$$

Ex. 404. If R denote the radius of a circle, and a one side of a regular inscribed octagon, to prove

$$a = R\sqrt{2 - \sqrt{2}}.$$



Let AB be the side of the regular inscribed octagon, and R the radius.

To prove $a = R\sqrt{2 - \sqrt{2}}$.

Proof. Draw AC, the side of an inscribed square.

$$OK = \tfrac{1}{2} R\sqrt{2}. \qquad \text{(Ex. 379)}$$

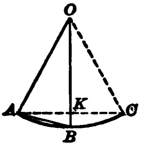
$$BK = OB - OK = R - \tfrac{1}{2}R\sqrt{2} = \frac{R}{2}(2 - \sqrt{2}).$$

Now $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2$, or $a^2 = \left(\frac{R}{2}\sqrt{2}\right)^2 + \left\{\frac{R}{2}(2-\sqrt{2})\right\}^2$. (Ex. 379)

$$\therefore a = R\sqrt{2-\sqrt{2}}.$$

Ex. 405. If R denote the radius of a circle, a the side of a regular inscribed dodecagon, to prove

$$a = R\sqrt{2-\sqrt{3}}.$$



Let AB be the side of a regular inscribed dodecagon, and R the radius.

To prove $a = R\sqrt{2-\sqrt{3}}$.

Proof. Draw AC, the side of a regular inscribed hexagon.

$$OK = \tfrac{1}{2}R\sqrt{3}. \quad \text{(Ex. 380)}$$

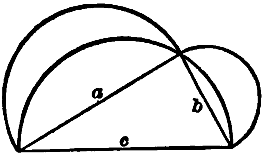
$$BK = OB - OK = R - \tfrac{1}{2}R\sqrt{3} = \frac{R}{2}(2-\sqrt{3}).$$

Now $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2$,

or $a^2 = (\tfrac{1}{2}R)^2 + \left\{\frac{R}{2}(2-\sqrt{3})\right\}^2$.

$$\therefore a = R\sqrt{2-\sqrt{3}}$$

Ex. 406. If on the legs of a right triangle, as diameters, semicircles are described external to the triangle, and from the whole figure a semicircle on the hypotenuse is subtracted, the remainder is equivalent to the given triangle.



Let a and b be the legs, and c the hypotenuse of the rt. △.

To prove the semicircles on a and b as diameters, increased by the area of the △ and diminished by the semicircle on c as a diameter, are equivalent to the area of the △.

Proof. Semicircles on a and b as diameters $= \tfrac{1}{8}\pi a^2 + \tfrac{1}{8}\pi b^2$

$$= \tfrac{1}{8}\pi(a^2+b^2) = \tfrac{1}{8}\pi c^2.$$

But semicircle on c as diameter $= \tfrac{1}{8}\pi c^2$.

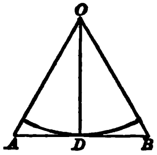
But, if the area of the △ be increased by $\tfrac{1}{8}\pi c^2$ and diminished by $\tfrac{1}{8}\pi c^2$, the triangle will remain.

Ex. 407. The radius of a circle $= r$. Find one side of the circumscribed equilateral triangle.

Solution. The side of the inscribed equilateral △ $= r\sqrt{3}$ (Ex. 378).

The side of the circumscribed △, being double the side of the inscribed △, is equal to $2r\sqrt{3}$ (Ex. 388).

Ex. 408. The radius of a circle is r. Find one side of the circumscribed regular hexagon.



Let AB, the side of a regular circumscribed hexagon, be denoted by a, and the radius of the ⊙ by r.

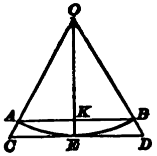
To find a in terms of r.

SOLUTION. OAB is an equilateral △.

$$AD = \sqrt{\overline{AO}^2 - \overline{OD}^2}, \text{ or } \frac{a}{2} = \sqrt{a^2 - r^2}.$$

$$\therefore a = \frac{2r}{3}\sqrt{3}.$$

Ex. 409. If the radius of a circle is r, and the side of an inscribed regular polygon is a, show that the side of the similar circumscribed regular polygon is $\frac{2ar}{\sqrt{4r^2 - a^2}}$.



Let AB, the side of a regular inscribed polygon, be denoted by a, and the radius of the ⊙ by r. Let CD be a side of the similar regular circumscribed polygon. (§ 407)

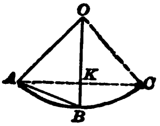
To prove $$CD = \frac{2ar}{\sqrt{4r^2 - a^2}}.$$

PROOF. $$OK = \sqrt{\overline{OA}^2 - \overline{AK}^2} = \sqrt{r^2 - \left(\frac{a}{2}\right)^2} = \tfrac{1}{2}\sqrt{4r^2 - a^2}.$$

But $$OK : OE :: AB : CD, \text{ or } \tfrac{1}{2}\sqrt{4r^2 - a^2} : r :: a : CD.$$

$$\therefore CD = \frac{2ar}{\sqrt{4r^2 - a^2}}$$

Ex. 410. The radius of a circle $= r$. Prove that the area of a regular inscribed octagon is $2r^2\sqrt{2}$.



Let AB be the side of a regular inscribed octagon, and r the radius of the ⊙.

To prove the area of the octagon $= 2r^2\sqrt{2}$.

PROOF. Draw AC, the side of the inscribed square.

$$AK = \frac{AO}{\sqrt{2}} \text{ (§ 340) } = \frac{r}{\sqrt{2}} = \tfrac{1}{2}r\sqrt{2}.$$

$$\triangle AOB = \tfrac{1}{2}AK \times BO \text{ (§ 368) } = \tfrac{1}{2} \times \tfrac{1}{2}r\sqrt{2} \times r = \tfrac{1}{4}r^2\sqrt{2}.$$

$$\text{The octagon} = 8\triangle AOB = 8 \times \tfrac{1}{4}r^2\sqrt{2} = 2r^2\sqrt{2}.$$

Ex. 411. The sides of three regular octagons are 3 feet, 4 feet, and 5 feet, respectively. Find the side of a regular octagon equal in area to the sum of the areas of the three given octagons.

SOLUTION. The side of a regular octagon equal in area to the first two

$$=\sqrt{3^2+4^2}=5 \text{ feet.} \qquad (\S\ 384)$$

The side of a regular octagon equal in area to all three is

$$\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2} \text{ feet.} \qquad (\S\ 384)$$

Ex. 412. What is the width of a ring between two concentric circumferences whose lengths are 440 feet and 330 feet?

SOLUTION. The diameters of the ⊙s are

$$\frac{440}{\pi} \text{ and } \frac{330}{\pi}.$$

The width of the ring between two concentric circumferences is one-half the difference between their diameters.

$$\frac{1}{2}\left(\frac{440}{\pi}-\frac{330}{\pi}\right)=\frac{55}{\pi}=17.5 \text{ feet.}$$

Ex. 413. Find the angle subtended at the centre by an arc 6 feet 5 inches long, if the radius of the circle is 8 feet 2 inches.

SOLUTION. 6 feet 5 inches = 77 inches; 8 feet 2 inches = 98 inches.

Circumference of ⊙ = $2\pi \times 98$ inches.

The part of the circumference subtended by an arc 77 inches long is

$\frac{77}{2\pi\times 98}$

$$\frac{77}{2\pi\times 98} \text{ of } 360^\circ=45.01^\circ.$$

Ex. 414. Find the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

SOLUTION. $$C=2\pi R. \qquad (\S\ 419)$$

$$\therefore R=\frac{C}{2\pi}.$$

That is, $$R=\frac{360^\circ}{2\times 3.1416}=57.296^\circ.$$

Ex. 415. What is the length of the arc subtended by one side of a regular dodecagon inscribed in a circle whose radius is 14 feet.

SOLUTION. Total circumference = $2\pi \times 14$ feet.

$$\tfrac{1}{12} \text{ of circumference}=\frac{2\pi\times 14}{12} \text{ feet}=7.33 \text{ feet.}$$

Ex. 416. Find the side of a square equivalent to a circle whose radius is 56 feet.

Solution. Area of $\odot = \pi \times 56^2$.

Side of equivalent square $= \sqrt{\pi \times 56^2} = 56\sqrt{\pi} = 99.25$ feet.

Ex. 417. Find the area of a circle inscribed in a square containing 196 square feet.

Solution. Diameter of $\odot$ = side of square $= \sqrt{196} = 14$ feet.

Area of $\odot = \frac{1}{4}\pi \times 14^2 = 49\pi = 153.938$ square feet.

Ex. 418. The diameter of a circular grass plot is 28 feet. Find the diameter of a circular plot just twice as large.

Solution. Since diameters of $\odot$s are to each other as the square roots of areas, the diameter of the second $\odot$ would be $28\sqrt{2} = 39.597$ feet.

Ex. 419. Find the side of the largest square that can be cut out of a circular piece of wood whose radius is 1 foot 8 inches.

Solution. The diameter of the $\odot$ is the diagonal of the inscribed square, and the side of a square is the diagonal divided by $\sqrt{2}$ (§ 340).

Radius = 1 foot 8 inches = $1\frac{2}{3}$ feet.

∴ diameter = $3\frac{1}{3}$ feet.

Side of square $= 3\frac{1}{3} \div \sqrt{2} = \frac{5}{3}\sqrt{2} = 2.357$ feet.

Ex. 420. The radius of a circle is 3 feet. What is the radius of a circle 25 times as large? $\frac{1}{9}$ as large? $\frac{1}{10}$ as large?

Solution. The radii required would be respectively

$\sqrt{25} \times 3$ feet = 15 feet; $\sqrt{\frac{1}{9}} \times 3$ feet = 1 foot; $\sqrt{\frac{1}{10}} \times 3$ feet = 0.948 feet.

Ex. 421. The radius of a circle is 9 feet. What are the radii of the concentric circumferences that will divide the circle into three equivalent parts.

Solution. The area of the whole $\odot = 81\pi$.

∴ the area of each part $= 27\pi$.

Since the area of a $\odot$ is equal to the square of the radius times π (§ 425), the radius of a $\odot$ is the square root of the area divided by π.

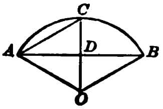
Hence the radius of the inner $\odot$ whose area is 27π would be

$$\sqrt{\frac{27\pi}{\pi}} = 3\sqrt{3} = 5.196 \text{ feet.}$$

Also, the radius of the second $\odot$ whose area is 54π would be

$$\sqrt{\frac{54\pi}{\pi}} = 3\sqrt{6} = 7.348 \text{ feet.}$$

Ex. 422. The chord of half an arc is 12 feet, and the radius of the circle is 18 feet. Find the height of the arc.



SOLUTION.

$$\overline{AC}^2 = 2\,OC \times CD. \quad (\S\ 337)$$
$$12^2 = 2 \times 18 \times CD.$$
$$36\,CD = 144,$$
$$CD = 4.$$

Ex. 423. The chord of an arc is 24 inches, and the height of the arc 9 inches. Find the diameter of the circle.

SOLUTION. Referring to the figure of Ex. 422, let K be the other extremity of the diameter passing through C.

Then $\overline{AD}^2 = CD \times DK$ (§ 337), or $12^2 = 9 \times DK.$

$\therefore DK = 16$, and $CK = DK + DC = 16 + 9 = 25$ inches.

Ex. 424. Find the area of a sector, if the radius of the circle is 28 feet, and angle at the centre $22\frac{1}{2}°$.

SOLUTION. $22\frac{1}{2}° \div 360° = \frac{1}{16}$. $\therefore$ the sector is $\frac{1}{16}$ of the ⊙.

$\therefore$ area of sector $= \frac{1}{16}$ of $28^2 \times \pi = 153.938.$

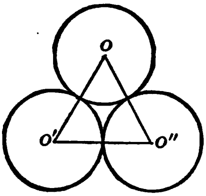
Ex. 425. The radius of a circle $= r$. Find the area of the segment subtended by one side of the inscribed regular hexagon.

SOLUTION. The area of the corresponding sector is $\frac{1}{6}$ of the ⊙, or $\frac{1}{6}\pi r^2$.

The area of the equilateral △ formed by the radii and the chord of the segment is $\frac{1}{4}r^2\sqrt{3}$ (Ex. 307).

$\therefore$ the area of the segment $= \frac{1}{6}\pi r^2 - \frac{1}{4}r^2\sqrt{3} = \frac{r^2}{12}(2\pi - 3\sqrt{3}).$

Ex. 426. Three equal circles are described, each touching the other two. If the common radius is r, find the area contained between the circles.



SOLUTION. The △ $OO'O''$ is equilateral, each side being $2r$, and the area $r^2\sqrt{3}$ (Ex. 307).

The area of the sector cut out of each ⊙ by the △ is $\frac{1}{6}$ of the ⊙, or $\frac{1}{6}\pi r^2$.

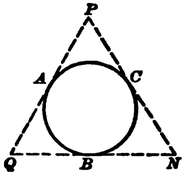
$\therefore$ the area of the three sectors is $\frac{1}{2}\pi r^2$.

But the area contained between the Ⓢ is the area of the △ diminished by the area of the three sectors; that is,

Area contained between the circles $= r^2\sqrt{3} - \frac{1}{2}\pi r^2 = \frac{1}{2}r^2(2\sqrt{3} - \pi).$

Ex. 427. To circumscribe about a given circle an equilateral triangle.

Let ABC be a ⊙. It is required to circumscribe about this ⊙ an equilateral △.



Construction. Find the points A, B, C, the vertices of an inscribed equilateral △ (§ 432). Through the points A, B, C draw tangents (§ 287) intersecting in Q, N, P. Then QNP is the △ required.

Proof. $\angle P$ is measured by $\frac{1}{2}$(arc ABC − arc AC) (§ 270)

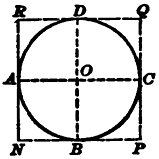
$= \frac{1}{2}$ of $\frac{1}{3}$ of the circumference $= \frac{1}{6}$ of the circumference $= 60°$.

In like manner $\angle N = 60° = \angle Q$.

∴ △ QNP is equiangular, and therefore equilateral. (§ 157)

Ex. 428. To circumscribe about a circle a square.

Let $ABCD$ be a ⊙. It is required to circumscribe a square about this ⊙.



Construction. Draw the diameters AC and $DB \perp$ to each other. Through the points A, B, C, D draw tangents intersecting in N, P, Q, R. Then $NPQR$ is the required square.

Proof. RQ and NP are $\perp$ to DB (§ 240), and are ∥ (§ 100).

RN and QP are $\perp$ to AC (§ 240) and are ∥. (§ 100)

∴ $NPQR$ is a ▱. (§ 168)

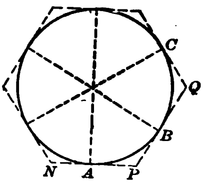
But $PQ = AC = DB = RN$. (§ 180)

DB is $\perp$ to AC (Cons.); and hence PQ is $\perp$ to PN.

∴ $NPQR$ is a square.

Ex. 429. To circumscribe about a circle a regular hexagon.

Let ABC be a ⊙. It is required to circumscribe about this ⊙ a regular hexagon.



Construction. Divide the circumference into 6 equal arcs at the points A, B, C, etc. (§ 431).

Through the points of division draw tangents intersecting in N, P, Q, etc. Then will NPQ.... be the required hexagon.

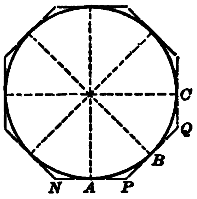
Proof. The $\angle P$ is measured by $\frac{1}{2}$(arc ACB − arc AB) [§ 270] $= \frac{1}{2}$ of $\frac{2}{3}$ of the circumference $= \frac{1}{3}$ of the circumference.

In like manner each of the other $\angle$s of the hexagon is measured by $\frac{1}{3}$ of the circumference.

Hence the hexagon is equiangular, and therefore regular (Ex. 397).

Ex. 430. To circumscribe about a circle a regular octagon.

Let ABC be a $\odot$. It is required to circumscribe a regular octagon.



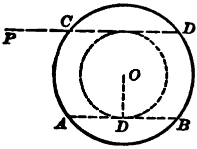
Construction. Divide the circumference into 8 equal parts (§ 430). Through the points of division draw tangents intersecting in N, P, Q, etc. Then NPQ..... is the octagon required.

Proof. $\angle P$ is measured by $\frac{1}{2}$(arc ACB – arc AB) [§ 270] $= \frac{1}{2}$ of $\frac{6}{8}$ of the circumference $= \frac{3}{8}$ of the circumference.

In like manner each of the other $\angle$s P, Q, etc., is measured by $\frac{3}{8}$ of the circumference.

Hence NPQ..... is equiangular, and therefore regular (Ex. 397).

Ex. 431. To draw through a given point a line so that it shall divide a given circumference into two parts having the ratio 3 : 7.



Let ABD' be a given $\odot$ and P a given point. It is required to construct a line through P which shall divide the circumference into two arcs having the ratio 3 : 7.

Construction. Divide the circumference into 10 equal parts (§ 434). Let the arc AB include three of these parts. Join AB. Let fall $OD \perp$ to AB. With O as a centre and OD as a radius describe a $\odot$. From the point P draw PCD' tangent to this $\odot$ (§ 287). Then will PD' divide the circumference as required.

Proof. chord CD' = chord AB. (§ 236)

$\therefore$ arc CD' = arc AB. (§ 230)

But arc $AB = \frac{3}{10}$ of the circumference.

$\therefore$ arc $CD' = \frac{3}{10}$ of the circumference,

and arc CD' : arc $CABD'$: : 3 : 7.

Ex. 432. To construct a circumference equal to the sum of two given circumferences.

Let C and C' be two given circumferences whose radii are respectively R and R'. It is required to construct a circumference equal to $C + C'$.

ANALYSIS. $C = 2\pi R$, and $C' = 2\pi R'$. (§ 419)

$$\therefore\ C + C' = 2\pi(R + R').$$

CONSTRUCTION. With $R + R'$ as a radius describe a circumference. This will be the circumference required.

PROOF. The circumference whose radius is $R + R'$ is

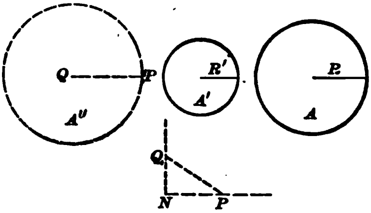
$$2\pi(R + R')\ (\S\ 419) = 2\pi R + 2\pi R' = C + C'.$$

Ex. 433. To construct a circle equivalent to the sum of two given circles.

Let A and A' be two ⊛ whose radii are R and R' respectively. It is required to construct a ⊙

$$A'' \Leftrightarrow A + A'.$$

CONSTRUCTION. Construct the rt. $\angle\ QNP$, and take $NP = R$ and $NQ = R'$. Join QP. Then with QP as a radius construct a ⊙ A''. Then will A'' be the ⊙ required.

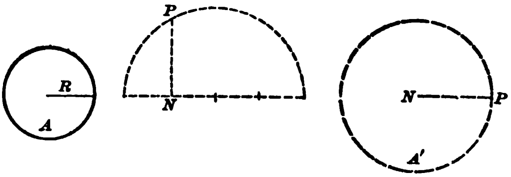


PROOF. $A = \pi R^2$ and $A' = \pi R'^2$. (§ 425)

$$\therefore\ A + A' = \pi(R^2 + R'^2) = \pi(\overline{NP}^2 + \overline{NQ}^2) = \pi\overline{QP}^2 = \text{area of ⊙ } A'',$$

whose radius is QP.

Ex. 434. To construct a circle equivalent to three times a given circle.



Let A be the given ⊙, whose radius is R. It is required to construct a ⊙ $A' \Leftrightarrow 3A$.

CONSTRUCTION. With R as a unit construct $NP = \sqrt{3}$ (Ex. 218). Then with NP as a radius describe a ⊙. This will be the ⊙ required.

PROOF. $A = \pi R^2$. (§ 425)

$$\therefore\ A' = \pi\overline{NP}^2 = \pi(R\sqrt{3})^2 = 3\pi R^2.$$

$$\therefore\ A' \Leftrightarrow 3A.$$

Ex. 435. To construct a circle equivalent to three-fourths of a given circle.

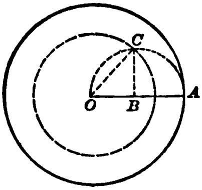
CONSTRUCTION. Construct $\sqrt{3}$ with the radius of the given ⊙ as a unit, as in Ex. 434. With one-half of this line as a radius describe a ⊙. This will be the ⊙ required.

PROOF. Let R be the given, and R' the required, radius, and A the given, and A' the required, ⊙.

Then $$\pi R'^2 = \tfrac{3}{4}\pi R^2.$$

$$\therefore\ R' = \frac{R\sqrt{3}}{2}.$$

Ex. 436. To divide a circle into two equivalent parts by a concentric circumference.



It is required to divide the ⊙ whose radius is OA into two equivalent parts by a concentric circumference.

CONSTRUCTION. Divide OA into two equal parts at B. Erect the ⊥ BC, and let it meet the semi-circumference OCA at C. Join OC. With O as a centre and OC as a radius describe a circumference. This will be the required line of division.

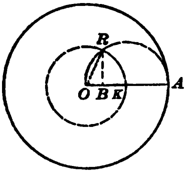
PROOF. $$\overline{OC}^2 = \overline{OB}^2 + \overline{BC}^2 = 2\,\overline{OB}^2 = \frac{4\,\overline{OB}^2}{2} = \frac{\overline{OA}^2}{2}.$$

$$\therefore\ \pi\overline{OC}^2 = \frac{\pi\overline{OA}^2}{2}.$$

But $\pi\overline{OC}^2$ is the area of the inner ⊙, (§ 425)

and $\dfrac{\pi\overline{OA}^2}{2}$ is $\tfrac{1}{2}$ of the area of the given circle.

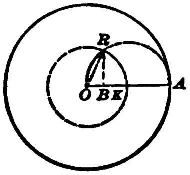
Ex. 437. To divide a circle into 5 equivalent parts by concentric circumferences.



It is required to divide the ⊙ of which OA is the radius into 5 equivalent parts by concentric circumferences.

CONSTRUCTION. Divide OA into 5 equal parts (§ 279). At B, the first point of division from O, erect a ⊥, and let it meet the semi-circumference on OA as a diameter in R. With O as a centre and OR as a radius, describe a circumference. This circumference will include $\tfrac{1}{5}$ of the ⊙.

If at K, the second point of division from O, another $\perp$ be erected, and a $\odot$ be described with O as a centre, and a radius equal to the distance from O to the intersection of the $\perp$ with the circumference on OA, this $\odot$ will include $\frac{2}{5}$ of the $\odot$, etc.



Proof. $\overline{OR}^2 = \overline{OB}^2 + \overline{BR}^2$ (§ 338)

$= \overline{OB}^2 + OB \times BA$ (§ 337)

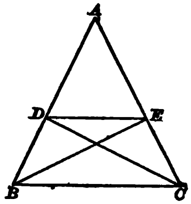
$= OB(OB + BA)$

$= \frac{1}{5} OA(OA) = \frac{1}{5}\overline{OA}^2.$

$\therefore \pi\overline{OR}^2 = \frac{1}{5}\pi\overline{OA}^2.$

But $\pi\overline{OR}^2$ is the area of the $\odot$ of which OR is the radius, and $\pi\overline{OA}^2$ is the area of the given $\odot$ (§ 425). In like manner it may be shown that $\pi\overline{OK}^2 = \frac{2}{5}\pi\overline{OA}^2$, etc.

Ex. 438. The line joining the feet of the perpendiculars dropped from the extremities of the base of an isosceles triangle to the opposite sides is parallel to the base.



Let ABC be an isosceles $\triangle$ with CD and BE $\perp$ to the equal sides AB and AC respectively.

To prove $DE \parallel$ to BC.

Proof. In the rt. $\triangle$s CBD and BCE,

$BC = BC$, (Iden.)

and $\angle CBD = \angle BCE$. (§ 154)

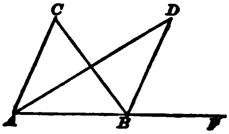
$\therefore \triangle CBD = \triangle BCE$. (§ 148)

$\therefore BD = CE$. $\therefore AB - BD = AC - CE$, or $AD = AE$.

That is, $\triangle ADE$ is isosceles; and since $\angle A$ is common to the $\triangle$s ADE and ABC,

$\therefore \angle ADE = \angle ABC$, and hence DE is $\parallel$ to BC. (§ 108)

Ex. 439. If AD bisects the angle A of a triangle ABC, and BD bisects the exterior angle CBF, then angle ADB equals one-half angle ACB.



Let AD bisect $\angle A$, and let BD bisect $\angle CBF$.

To prove $\angle D = \frac{1}{2}\angle C$.

Proof. $\angle CBF = \angle CAB + \angle C$. (§ 145)

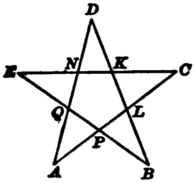
$\therefore \angle DBF = \angle DAB + \frac{1}{2}\angle C$.

But $\angle DBF = \angle DAB + \angle D$. (§ 145)

$\therefore \angle DAB + \frac{1}{2}\angle C = \angle DAB + \angle D$. (Ax. 1)

$\therefore \frac{1}{2}\angle C = \angle D$. (Ax. 3)

Ex. 440. The sum of the acute angles at the vertices of a pentagram is equal to two right angles.



Let A, B, C, D, E be the acute ∡ at the vertices of the pentagram.

To prove $A + B + C + D + E = 2$ rt. ∡.

Proof. $\angle DNK + \angle CKL + \angle BLP$
$+ \angle APQ + \angle EQN = 4$ rt. ∡. (§ 207)

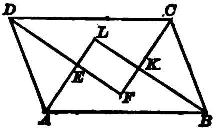
Also $\angle DKN + \angle CLK + \angle BPL$
$+ \angle AQP + \angle ENQ = 4$ rt. ∡. (§ 207)

Hence sum of base ∡ of the ▲ DNK, CKL, etc. $=$ 8 rt. ∡. (Ax. 2)

But sum of all the ∡ of the ▲ DNK, CKL, etc. $=$ 10 rt. ∡. (§ 138)

$\therefore A + B + C + D + E = 2$ rt. ∡. (Ax. 3)

Ex. 441. The bisectors of the angles of a parallelogram form a rectangle.



Let $ABCD$ be a parallelogram. Let AL, BL, CF, DF bisect respectively ∡ A, B, C, D, and meet in E, F, K, L.

To prove $EFKL$ a rectangle.

Proof. $\angle CDA + \angle DAB = 2$ rt. ∡. (§ 109)

$\therefore \angle EDA + EAD = 1$ rt. $\angle$. (Ax. 7)

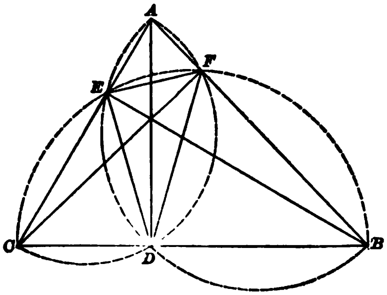
$\therefore \angle AED = 1$ rt. $\angle$. (§ 138)

$\therefore \angle FEL = 1$ rt. $\angle$. (§ 95)

In like manner L, K, F may be proved rt. ∡.

$\therefore EFKL$ is a rectangle.

Ex. 442. The altitudes AD, BE, CF of the triangle ABC bisect the angles of the triangle DEF.



Let ED, EF, FD join the feet of the altitudes of the $\triangle$ ABC.

To prove

$\angle EDA = \angle FDA$,

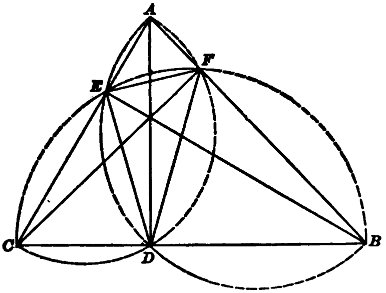
$\angle FEB = \angle DEB$,

$\angle EFC = \angle DFC$.

Proof. On AB, BC, AC as diameters describe ⓢ. These ⊙ will pass through D and

E, *E* and *F*, *D* and *F* respectively (since *ADB* and *AEB*, *CEB* and *CFB*, *AFC* and *ADC* are right angles).

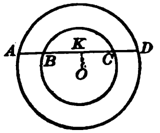
$\angle EDA = \angle EBA$ (each being measured by ½ arc *EA*) $= \angle ECF$ (each being measured by ½ arc *EF*) $= \angle FDA$ (each being measured by ½ arc *AF*).



Also, $\angle FEB = \angle FCB$ (each being measured by ½ arc *FB*) $= \angle DAF$ (each being measured by ½ arc *DF*) $= \angle DEB$ (each being measured by ½ arc *DB*).

Also, $\angle EFC = \angle EBC$ (each being measured by ½ arc *EC*) $= \angle EAD$ (each being measured by ½ arc *ED*) $= \angle DFC$ (each being measured by ½ arc *CD*).

Ex. 443. The portions of any straight line intercepted between the circumferences of two concentric circles are equal.



Let *AD* cut the outer ⊙ in *A* and *D* and the inner ⊙ in *B* and *C*.

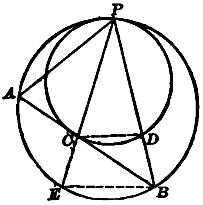
To prove $AB = DC$.

Proof. Draw *OK* from the centre *O* ⊥ to *AD*.

Then $KA = KD$ and $KB = KC$. (§ 232)

$\therefore AB = DC$. (Ax. 3)

Ex. 444. Two circles are tangent internally at *P*, and a chord *AB* of the larger circle touches the smaller circle at *C*. Prove that *PC* bisects the angle *APB*.



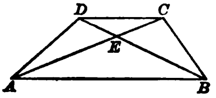
Let the ⊛ *PCD* and *PAB* be tangent internally at *P*. Let *AB* be a chord of the larger ⊙ *PAB* and a tangent to *PCD* at *C*.

To prove $\angle APC = \angle BPC$.

Proof. Draw *CD* and *EB*. *CD* is ∥ to *EB* (Ex. 237, § 312).

$\angle APC = \angle ABE$ (each being measured by ½ arc *AE*) $= \angle BCD$ (§ 104) $= \angle BPC$, since ½ arc *CD* measures $\angle BPC$ (§ 263) and $\angle BCD$ (§ 269).

Ex. 445. The diagonals of a trapezoid divide each other into segments which are proportional.



Let $ABCD$ be a trapezoid whose diagonals AC and BD intersect in E.

To prove $EA : EB = EC : ED$.

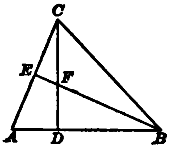
Proof. In the $\triangle$ AEB and DEC, $\angle AEB = \angle DEC$. (§ 95)

$\angle EBA = \angle EDC$. (§ 104)

$\therefore$ $\triangle$ AEB and DEC are similar. (§ 322)

$\therefore EA : EB = EC : ED$.

Ex. 446. The perpendiculars from two vertices of a triangle upon the opposite sides divide each other into segments reciprocally proportional.



Let ABC be a $\triangle$. Let CD and BE be $\perp$ respectively to AB and AC, and meeting in F.

To prove $FB : FC = FD : FE$.

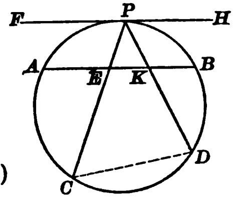
Proof. In the rt. $\triangle$ CEF and BFD, $\angle CFE = \angle DFB$. (§ 95)

$\therefore$ the $\triangle$ are similar. (§ 323)

$\therefore FB : FC = FD : FE$.

Ex. 447. If through a point P in the circumference of a circle two chords are drawn, the whole chords and the segments between P and a chord parallel to the tangent at P are reciprocally proportional.

Let the chord AB be $\parallel$ to the tangent at P, and let PC and PD be chords cutting AB in E and K.



To prove $PE : PK = PD : PC$.

Proof. Join CD.

$\angle FPE = \angle D$,

(since each is measured by $\frac{1}{2}$ arc PC).

But $\angle FPE = \angle PEK$. (§ 104)

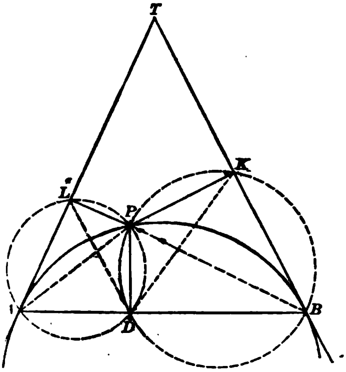
$\therefore \angle PEK = \angle D$.

In like manner $\angle PKE = \angle C$.

$\therefore$ $\triangle$ PEK and PCD are similar. (§ 321)

$\therefore PE : PK = PD : PC$.

Ex. 448. The perpendicular from any point of a circumference upon a chord is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.



Let AB be a chord, AT and BT tangents, PD drawn from any point P in the arc $\perp$ to AB. Let PL and PK be $\perp$ to AT and BT respectively.

To prove $PL : PD = PD : PK$.

Proof. On BP and AP as diameters, describe ⊙s. These ⊙s will pass through K, D and L, D respectively, since PKB, PDB, ALP, and ADP are rt. $\angle$s. Join LD and KD.

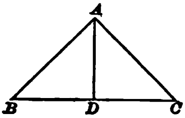
$\angle PKD = \angle PBD$ (§ 263) $= \angle LAP$ (since $\frac{1}{2}$ arc AP measures $\angle ABP$ and $\angle LAP$) $= \angle LDP$ (§ 263).

Also, $\angle PLD = \angle PAD$ (§ 263) $= \angle KBP$ (since $\frac{1}{2}$ arc BP measures $\angle PAB$ and $\angle PBK$) $= \angle KDP$ (§ 263).

$\therefore$ $\triangle$s LPD and KPD are similar. (§ 322)

$\therefore$ $PL : PD = PD : PK$.

Ex. 449. In an isosceles right triangle either leg is a mean proportional between the hypotenuse and the perpendicular upon it from the vertex of the right angle.



Let BAC be an isosceles $\triangle$ right-angled at A. Let AD be perpendicular to BC.

To prove $BC : AB = AB : AD$.

Proof. $BC : AB = AB : BD$. (§ 334)

But $\angle B = \angle DAB = 45°$.

$\therefore$ $BD = AD$. (§ 156)

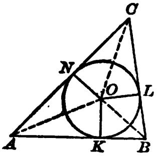
$\therefore$ $BC : AB = AB : AD$.

Ex. 450. The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.

Let ABC be a $\triangle$, and NLK be the inscribed ⊙ having the radius OK.

To prove . area $\triangle = \frac{1}{2}(AB + BC + CA) \times OK$.

Proof. The $\perp$s OK, OL, ON are all equal (§ 162).



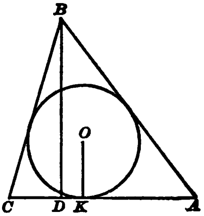
$$\triangle AOB = AB \times \tfrac{1}{2} OK,$$
$$\triangle BOC = BC \times \tfrac{1}{2} OL,$$
$$\triangle AOC = AC \times \tfrac{1}{2} ON.$$

By adding these equations,

$$\triangle AOB + \triangle BOC + \triangle AOC = \tfrac{1}{2}(AB + BC + AC) \times OK.$$
$$\therefore \triangle ABC = \tfrac{1}{2}(AB + BC + AC) \times OK.$$

Ex. 451. The perimeter of a triangle is to one side as the perpendicular from the opposite vertex is to the radius of the inscribed circle.

Let ABC be a $\triangle$ whose perimeter is p. Let BD be $\perp$ to CA, and OK the radius of the inscribed $\odot$.



To prove $p : AC = BD : OK$.

Proof. area $\triangle = \frac{1}{2} p \times OK$. (Ex. 450)

Also, area $\triangle = \frac{1}{2} AC \times BD$. (§ 368)

$\therefore \frac{1}{2} p \times OK = \frac{1}{2} AC \times BD$. (Ax. 1)

$\therefore p \times OK = AC \times BD$. (Ax. 6)

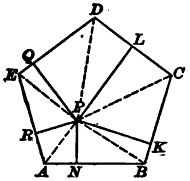
$\therefore p : AC = BD : OK$. (§ 297)

Ex. 452. The sum of the perpendiculars from any point within an equilateral convex polygon upon the sides is constant.

Let $ABCDE$ be an equilateral convex polygon, and PN, PK, PL, PQ, PR the $\perp$s from any point P within the polygon upon the sides.

To prove $PN + PK + PL + PQ + PR$ constant.

Proof. Join PA, PB, PC, PD, PE.



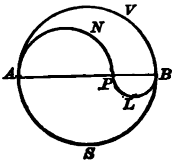
$$\triangle PAB = \tfrac{1}{2} AB \times PN.$$
$$\triangle PBC = \tfrac{1}{2} BC \times PK.$$
$$\triangle PCD = \tfrac{1}{2} CD \times PL.$$
$$\triangle PDE = \tfrac{1}{2} DE \times PQ.$$
$$\triangle PEA = \tfrac{1}{2} EA \times PR.$$

Adding these equations together, and noting that the sum of these $\triangle$s is the area of the polygon and that each side of the polygon $= AB$, we have

$$\text{area of polygon} = \tfrac{1}{2} AB(PN + PK + PL + PQ + PR).$$

But area of polygon and $\frac{1}{2} AB$ are constant, and hence the sum of the perpendiculars is constant.

Ex. 453. A diameter of a circle is divided into any two parts, and upon these parts as diameters semi-circumferences are described on opposite sides of the given diameter. Prove that the sum of their lengths is equal to the semi-circumference of the given circle, and that they divide the circle into two parts, whose areas have the same ratio as the two parts into which the diameter is divided.



Let AB be the diameter of the ⊙. Let P be any point on AB, at distance x from the centre, so that $AP = r + x$ and $PB = r - x$; upon AP and PB as diameters let semicircles be described on opposite sides of AB.

I. To prove curved line $ANPLB = \pi r$.

PROOF. $\text{arc } ANP = \frac{1}{2}(r + x)\pi.$ (§ 419)

$\text{arc } PLB = \frac{1}{2}(r - x)\pi.$ (§ 419)

$$\therefore \text{curve } ANPLB = \tfrac{1}{2}(r + x)\pi + \tfrac{1}{2}(r - x)\pi = \pi r.$$

II. To prove $\dfrac{\text{area } ANPLBS}{\text{area } AVBLPN} = \dfrac{r + x}{r - x}.$

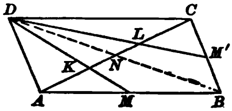
PROOF.

$$\frac{\text{area } ANPLBS}{\text{area } AVBLPN} = \frac{\text{semicircle } ANP + \text{semicircle } ASB - \text{semicircle } PLB}{\text{semicircle } AVB + \text{semicircle } PLB - \text{semicircle } ANP}$$

$$= \frac{\frac{1}{8}(r + x)^2\pi + \frac{\pi r^2}{2} - \frac{1}{8}(r - x)^2\pi}{\frac{\pi r^2}{2} + \frac{1}{8}(r - x)^2\pi - \frac{1}{8}(r + x)^2\pi} = \frac{r + x}{r - x}.$$

Ex. 454. Lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect one of the diagonals.

Let AC be a ▱. Let M and M' be the middle points of AB and BC respectively. Let DM and DM' be drawn, cutting AC in K and L respectively.



To prove $AK = KL = LC.$

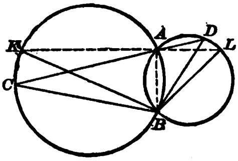
PROOF. Join DB. Since N is the middle of DB (§ 184), and M' is the middle of BC (hyp.), $LN = \frac{1}{3} CN$ (Ex. 33) $= \frac{1}{6} AC.$ In like manner, $KN = \frac{1}{6} AC.$

$$\therefore KL = \tfrac{1}{6} AC + \tfrac{1}{6} AC = \tfrac{1}{3} AC.$$

But $CL = 2 LN = \frac{1}{3} AC$, and $AK = 2 KN = \frac{1}{3} AC.$ $\therefore AK = KL = LC.$

Ex. 455. If two circles intersect in the points A and B, and through A any secant CAD is drawn limited by the circumferences at C and D, the straight lines BC, BD, are to each other as the diameters of the circles.

Let the two Ⓢ intersect in the points A and B, and CAD be drawn, cutting the circumferences in C and D. Let BC and BD be drawn, also the diameters BK and BL.



To prove $BC : BD = BK : BL$.

PROOF. Join AK and AL.

$\angle KAB$ and $\angle LAB$ are rt. $\angle$. (§ 264)

$\therefore AK$ and AL are in the same straight line (§ 93).

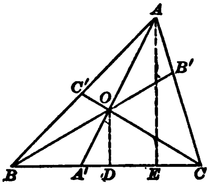
In the $\triangle$ CBD and KBL, $\angle C = \angle K$ (each being measured by ½ the same arc); also $\angle D = \angle L$ (each being measured by ½ the same arc).

$\therefore$ $\triangle$ CBD and KBL are similar. (§ 322)

$$\therefore BC : BD = BK : BL.$$

Ex. 456. If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$



Let O be any point in the $\triangle ABC$. Let AA', CC', BB', be drawn through O, cutting the sides BC, AB, AC in A', C', B', respectively.

To prove $$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

PROOF. Draw OD and $AE \perp$ to BC. By similarity of $\triangle$,

$$\frac{OD}{AE} = \frac{OA'}{AA'}.$$

$$\frac{\triangle OBC}{\triangle ABC} = \frac{OD}{AE} (\S\ 370) = \frac{OA'}{AA'}.$$

In like manner

$$\frac{\triangle OAC}{\triangle ABC} = \frac{OB'}{BB'}, \text{ and } \frac{\triangle OAB}{\triangle ABC} = \frac{OC'}{CC'}.$$

By adding these equations,

$$\frac{\triangle OBC + \triangle OAC + \triangle OAB}{\triangle ABC} = \frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'}.$$

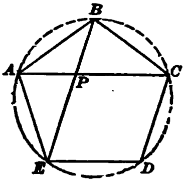
But $$\frac{\triangle OBC + \triangle OAC + \triangle OAB}{\triangle ABC} = \frac{\triangle ABC}{\triangle ABC} = 1.$$

$$\therefore \frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

Ex. 457. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.

Let BE and AC be diagonals of a regular pentagon and intersect at P.

To prove $AC : PC = PC : AP$.



Proof. Circumscribe a ⊙ about the pentagon. In the ▵ ABP and ABC, $\angle A$ is common, $\angle ABP = \angle ACB$ (each being measured by $\frac{1}{10}$ of the circumference).

$\therefore$ ▵ ABP and ABC are similar. (§ 322)

$\therefore AC : AB = AB : AP$.

$\angle CPB$, measured by $\frac{1}{2}$ (arc AE + arc BC) (§ 268), $= \frac{1}{5}$ of the circumference, and $\angle CBP$, measured by $\frac{1}{2}$ arc CDE, $= \frac{1}{5}$ of the circumference.

$$\therefore \angle CPB = \angle CBP.$$

$$\therefore PC = BC = AB.$$

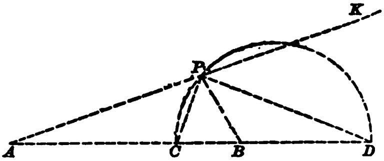
Substituting in the above proportion PC for its equal AB, we have

$$AC : PC : : PC : AP.$$

Ex. 458. Find the locus of a point P whose distances from two given points A and B are in a given ratio ($m : n$).

Let A and B be the two given points, and $m : n$ the given ratio. Let P be taken so that $PA : PB : : m : n$.

To find the locus of P.



Analysis. Suppose the construction effected, and P to be a point in the locus required.

Divide AB internally at C and externally at D, in the ratio $m : n$. (§§ 313, 314)

Then $PA : PB = m : n = CA : CB = AD : BD$.

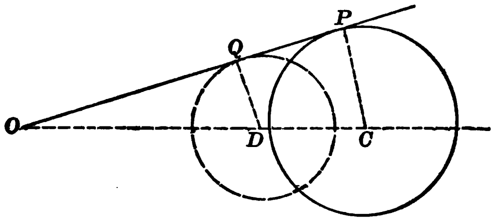
Now PC bisects the $\angle APB$, and PD bisects the $\angle BPK$.

$\therefore CPD$ is a rt. $\angle$. (Ex. 11)

$\therefore P$ is in the circumference on CD as a diameter (§ 264), and the required locus is this circumference.

Construction. Divide AB harmonically at C and D in the ratio $m : n$ (§ 317), and upon CD as a diameter describe a circumference. This circumference will be the required locus.

Ex. 459. OP is any straight line drawn from a fixed point O to the circumference of a fixed circle; in OP a point Q is taken such that OQ is to OP in a fixed ratio. Find the locus of Q.



Solution. Join OC, and in OC take D such that

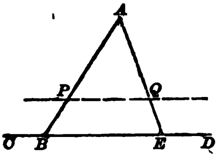
$$OD : OC = m : n \text{ (the given ratio).}$$

Join DQ and CP. $\triangle$ OCP and ODQ are similar. (§ 312)

$$\therefore DQ : CP = OD : OC = m : n.$$

But CP is constant; therefore DQ is constant. Hence the circumference of a ⊙ of which D is the centre and DQ the radius is the locus required.

Ex. 460. From a fixed point A a straight line AB is drawn to any point in a given straight line CD, and then divided at P in a given ratio ($m : n$). Find the locus of the point P.



Solution. Let AB be drawn from the fixed point A to meet CD. Let P be taken so that $AP : PB :: m : n$.

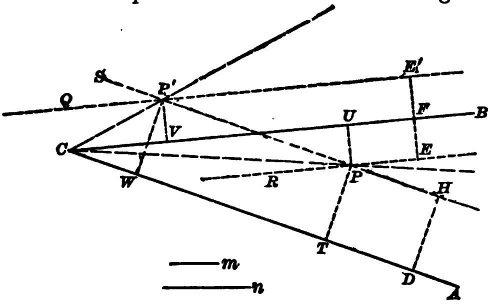
To find the locus of P.

Draw any other line through A, cutting CD in any point E. Through P draw $PQ \parallel$ to CD, cutting AE in Q.

Then $AP : PB = m : n = AQ : QE$. (§ 309)

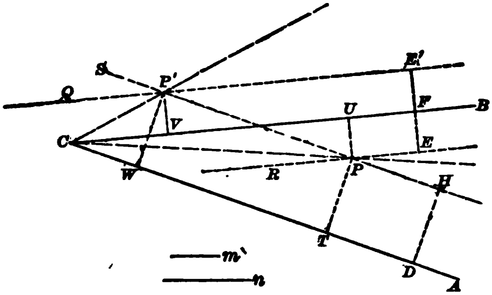
$$\therefore PQ \text{ is the locus required.}$$

Ex. 461. Find the locus of a point whose distances from two given straight lines are in a given ratio.



Let CA and CB be two lines, and $m : n$ the given ratio. It is required to find the locus of a point whose distances from CB and CA are in the ratio $m : n$.

Construction. From any point on CA, as D, draw $DH = n$, and $\perp$ to CA. Through H draw $HS \parallel$ to AC. From any point on CB, as F, draw $FE = m$, and $\perp$ to CB, and also $FE' = m$, and $\perp$ to CB. Through E draw $ER \parallel$ to BC, and intersecting HS in P, and through E' draw $E'Q \parallel$ to BC, and intersecting HS in P'. Then will the locus required consist of the two straight lines CP and CP'.

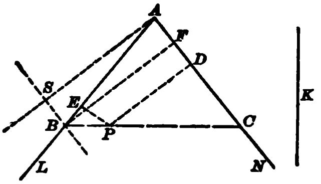


Proof. Since PT ($\perp$ to CA) $= HD = n$, and PU ($\perp$ to CB) $= EF = n$, the point P is a point of the locus required. Any other point on CP will also satisfy the required condition; for $\perp$s from this point on CB and CA will, by similarity of $\triangle$s, be as $m : n$. Therefore CP is one branch of the required locus. In like manner CP' may be shown to be another branch.

Ex. 462. Find the locus of a point the sum of whose distances from two given straight lines is equal to a given length K.

Let AL and AN be two given lines, and K the given length. It is required to find the locus of P so that

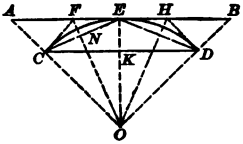
$$PE + PD = K.$$



Proof. Draw $AS = K \perp$ to AN. Through S draw $SB \parallel$ to AN, and cutting AL in B. Take $AC = AB$. Join BC. Then will BC be the locus required. For ABC is an isosceles $\triangle$ (Cons.), and $BF = K$. Hence if any point P be taken on BC, and $\perp$s PE, PD be drawn to AB, AC respectively, $PE + PD = BF = K$ (Ex. 73).

Ex. 463. Given the perimeters of a regular inscribed and a similar circumscribed polygon, to compute the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.

Let AB be one side of the circumscribed polygon, and $CD \parallel$ to AB, one side of the similar inscribed polygon. It is required to construct circumscribed and inscribed regular polygons of double the number of sides and to compute their perimeters in terms of the perimeters of the given polygons.



CONSTRUCTION. Draw $OE \perp$ to AB. Join AO, BO. Join CE, DE. Draw $CF \perp$ to AO, and $DH \perp$ to BO. Then will FH be a side of the circumscribed polygon required, and CE and DE sides of the inscribed polygon required.

PROOF. C and D lie in AO and BO respectively. (§ 407)

$$\text{arc } CE = \text{arc } DE. \qquad (§\ 232)$$

$$\therefore CE = DE,$$

and it is evident that CE and DE are sides of the required inscribed polygon. Then FH must be a side of the required circumscribed polygon (§ 406).

COMPUTATION. Let P and p respectively denote the perimeters of the given circumscribed and inscribed polygons, and P' and p' respectively denote the perimeters of the circumscribed and inscribed polygons of double the number of sides.

$$\frac{P}{p} = \frac{OA}{OC}\ (§\ 413) = \frac{OA}{OE}.$$

But in the two rt. △ OFC and OFE,

$$FO = FO \text{ and } OC = OE.$$

∴ the △ are equal, and FO bisects the ∠ COE.

$$\therefore \frac{OA}{OE} = \frac{AF}{EF}. \qquad (§\ 313)$$

$$\therefore \frac{P}{p} = \frac{AF}{EF}. \qquad (\text{Ax. } 1)$$

Whence, by composition, $\dfrac{P+p}{p} = \dfrac{AF+EF}{EF}$, (§ 300)

or $\dfrac{P+p}{2p} = \dfrac{AE}{2EF} = \dfrac{AE}{FH}$.

But AE is contained as often in P as FH is in P',

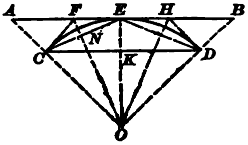
$$\therefore \frac{AE}{FH} = \frac{P}{P'}. \qquad (§\ 307)$$

$$\therefore \frac{P+p}{2p} = \frac{P}{P'}. \qquad \text{(Ax. 1)}$$

Whence $$P' = \frac{2pP}{P+p}.$$

Also, the rt. △ *CKE* and *EFN* are similar. (§ 323)

$$\therefore \frac{CK}{CE} = \frac{EN}{FE}.$$



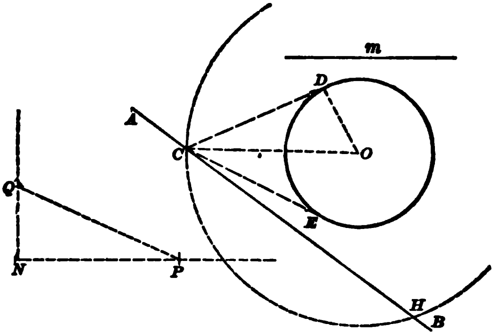
But $$\frac{CK}{CE} = \frac{p}{p'}, \text{ (§ 307)}$$

and $$\frac{EN}{FE} = \frac{p'}{P'}. \text{ (§ 307)}$$

$$\therefore \frac{p}{p'} = \frac{p'}{P'}. \qquad \text{(Ax. 1)}$$

Whence $$p' = \sqrt{p \times P'}.$$

Ex. 464. To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given straight line shall have a given length.



Let *DE* be the given ⊙, *m* the given length, and *AB* the given line. It is required to find a point on *AB* from which a tangent to the ⊙ shall be of the given length *m*.

ANALYSIS. Suppose *C* to be the required point on *AB*, and *CD* the required tangent. Join the centre *O* with *C* and *D*.

Then $\overline{OC}^2 = \overline{OD}^2 + \overline{CD}^2$ (§ 338) $= \overline{OD}^2 + m^2$.

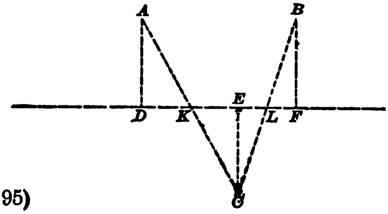
CONSTRUCTION. Draw NP at rt. $\triangle$ to NQ, and take $NP = m$ and $NQ = OD$. Join QP. With O as a centre and QP as a radius describe a ⊙ cutting AB in C. From C draw the tangent CD. This will be the tangent required.

PROOF. $CD = \sqrt{\overline{CO}^2 - \overline{OD}^2} = \sqrt{\overline{QP}^2 - \overline{NQ}^2} = NP = m.$

DISCUSSION. From C another tangent CE may be drawn which will fulfil the required conditions. Two other tangents from H may also be drawn. So that there are in general *four* tangents possible, two from each of the points in which the constructed ⊙ cuts AB. It is evident that if the line AB is at the distance OC from the centre, the two points C and H coincide, and but two tangents may be drawn; and if the line AB is at a greater distance from the centre than OC, the constructed ⊙ does not meet AB and there is no solution.

Ex. 465. To draw a straight line equidistant from three given points.

Let A, B, C be three given points. It is required to draw a line equidistant from them.



ANALYSIS. Let DF be the required line, and AD, BF, and CE the equal $\perp$s. Join AC and BC.

In the two rt. $\triangle$ ADK and CEK, $AD = CE$, (Hyp.)

and $\angle AKD = \angle CKE$. (§ 95)

$$\therefore \triangle ADK = \triangle CEK, \quad (§ 149)$$

and $AK = CK$.

In like manner $BL = CL$.

CONSTRUCTION. Join AC and BC. Bisect AC and BC in K and L. Through K and L draw DF. Then will DF be the required line.

PROOF. In the rt. $\triangle$ ADK and CEL,

$$AK = CK, \quad \text{(Cons.)}$$

and

$$\angle AKD = \angle CKE.$$

$$\therefore \triangle ADK = \triangle CEK. \quad (§ 148)$$

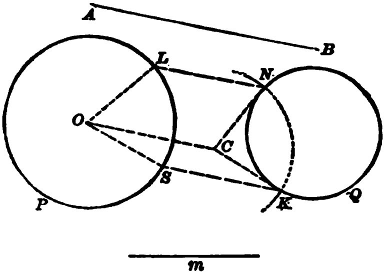
$$\therefore AD = CE.$$

In like manner $CE = BF$.

DISCUSSION. If the three given points are in a straight line, the required line would be ∥ to the line in which the points lie.

Ex. 466. To inscribe a straight line of given length between two given circumferences and parallel to a given straight line.

Let LP and NQ be two given ⊙s, AB the given direction, and m the given length. It is required to draw a line ∥ to AB, and so that a given length m shall be intercepted between the two circles.



Analysis. Suppose LN to be the line required. Join OL, and draw $OC = m$, and ∥ to LN. Join CN.

Since $LN = OC$, and LN is ∥ to OC, $OCNL$ is a ▱ (§ 182). ∴ $CN = OL$ and CN is ∥ to OL.

Construction. Draw $OC = m$, and ∥ to AB. With C as a centre and a radius equal to the radius of ⊙ LP, describe an arc, cutting ⊙ NQ in N. Draw OL ∥ to CN, and cutting ⊙ LP in L. Join LN. Then will LN be the line required.

Proof. Since CN is = and ∥ to OL (Cons.), $OCNL$ is a ▱. (§ 182)

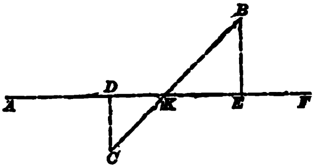
∴ LN is = and ∥ to OC.

But OC is = to m and ∥ to AB. ∴ $LN = m$, and is ∥ to AB.

Discussion. Since, in general, the arc NK cuts the ⊙ NQ in another point K, a second line SK will fulfil the required conditions. If arc NK touch ⊙ NQ, LN and SK will fall together, and only one construction is possible. If arc NK does not touch ⊙ NQ, the problem is impossible.

Ex. 467. To draw through a given point a straight line so that its distances from two other given points shall be in a given ratio.

Let A be the given point through which the straight line is to be drawn, and B and C the other given points, and $m : n$ the given ratio. It is required to draw AF so that the ⊥s from B and C shall be in the ratio $m : n$.



Analysis. Suppose AF to be the line required, and BE and CD be ⊥ to AF. Join BC, and let BC cut AF in K. In the rt. △s CKD and BKE, the ∠ CKD = ∠ BKE (§ 95); hence these △s are similar, and

$$BK : CK = BE : CD = m : n.$$

Construction. Join BC. Divide BC in the ratio $m:n$ (§ 351) at K. Through A and K draw AF. AF will be the required line.

Proof. The rt. $\triangle$s CKD and BKE are similar. (§ 323)

$$\therefore BE:CD=BK:CK=m:n.$$

Ex. 468. To construct a square equivalent to the sum of a given triangle and a given parallelogram.

Construction. Construct a square equivalent to the $\triangle$, by finding a mean proportional between the base and one-half the altitude of the given $\triangle$ (§ 390). Construct a square equivalent to the $\square$, by finding a mean proportional between the base and altitude of the $\square$ (§ 389).

Then construct a square equivalent to the sum of these two squares (§ 381).

Ex. 469. Construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.

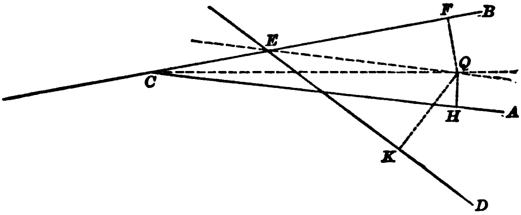
Construction. Construct a square equivalent to the given $\triangle$ (§ 390). Construct a square equivalent to the given pentagon (§ 391). Construct a square equivalent to the sum of these constructed squares (§ 381).

Finally, construct a rectangle equivalent to this last square and having the difference of its base and altitude the given line (§ 393).

Ex. 470. Construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.

This is a special case of § 394.

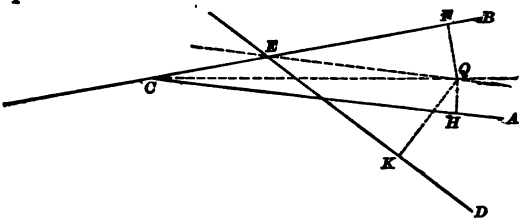
Ex. 471. To find a point whose distances from three given straight lines shall be as the numbers m, n, and p.



Let CA, CB, ED be the three lines. It is required to find a point Q such that its distances to the three lines in order shall be as m, n, p.

Construction. Construct CQ, the locus of a point whose distances from CA and CB are as $m : n$ (Ex. 461).

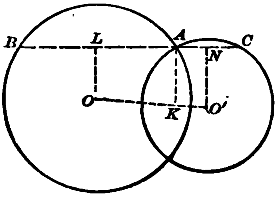
Construct EQ, the locus of a point whose distances from CB and ED are as $n : p$.



Let CQ and EQ intersect in Q. Then will Q be the point required.

Proof. Draw $QH \perp$ to CA, $QF \perp$ to CB, and $QK \perp$ to ED. By construction, QH, QF, QK are as the numbers m, n, p (Ex. 461).

Ex. 472. Given two circles intersecting in the point A. To draw through A a secant BAC such that AB shall be to AC in a given ratio $(m : n)$.



Let the $\circledS$ whose centres are O and O' intersect in A. It is required to draw BAC so that $BA : AC :: m : n$.

Analysis. Let BAC be the required line, and OO' the line of centres. Draw OL and $O'N \perp$ to BAC, and also $AK \perp$ to BAC.

Then $\qquad OL, KA, O'N$ are $\parallel$. $\qquad$ (§ 100)

$$\therefore OK : KO' = LA : AN = AB : AC = m : n \qquad (\text{§ } 311)$$

(since $AB = 2AL$ and $AC = 2AN$).

Construction. Join OO', and divide OO' in the ratio $m : n$ (§ 351) at K. Join AK. Through A draw $BAC \perp$ to AK. Then will BAC be the line required.

Proof. $\qquad$ Draw the $\perp$s OL, $O'N$ to BC.

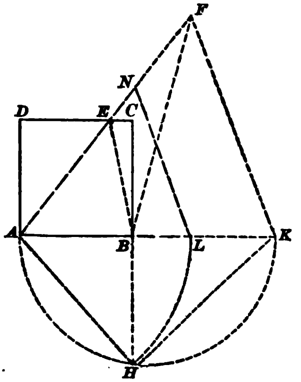
Then $\qquad OL, KA, O'N$ are $\parallel$. $\qquad$ (§ 100)

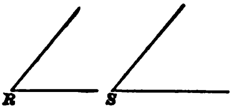
$$\therefore m : n = OK : KO' = LA : AN = AB : AC \qquad (\text{§ } 311)$$

(since by § 232 OL and $O'N$ bisect AB and AC respectively).

Ex. 473. To construct a triangle, given its angles and its area.

Let BD be a given square, and R and S given $\angle$s. It is required to construct a $\triangle$ having an area of BD, and R and S as two of its $\angle$s.





Construction. Draw AF, making $\angle FAB = \angle R$. Let AF cut DC in E. Take $EF = AE$, and join FB. Draw FK, making the $\angle AFK = \angle S$, and cutting AB produced in K. On AK as a diameter describe a semicircle, cutting CB produced in H. Join AH. Take AL on AK equal to AH. Through L draw $LN \parallel$ to KF. Then will $\triangle ANL$ be the $\triangle$ required.

Proof. $\triangle ANL$ has the required $\angle$s, since $R = \angle NAL$ and $\angle ANL = \angle AFK$ (§ 106) $= \angle S$.

Draw EB. $\triangle AEB \Bumpeq \frac{1}{2}$ square BD (§ 368).

Now, $\triangle AFB \Bumpeq 2\triangle AEB$, since these $\triangle$s have the same altitude, and the base $AF = 2AE$ (Cons.).

$$\therefore \triangle AFB \Bumpeq \text{square } BD.$$

Again, $$\overline{AH}^2 = \overline{AL}^2 = AB \times AK. \quad \text{(§ 334, II.)}$$

$$\triangle AKF : \triangle ALN = \overline{AK}^2 : \overline{AL}^2. \quad \text{(§ 375)}$$

$$\therefore \overline{AK}^2 : AB \times AK = AK : AB = \triangle AFK : \triangle AFB. \quad \text{(§ 370)}$$

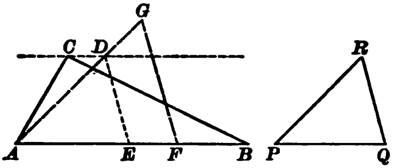
That is, $$\triangle AKF : \triangle ALN = \triangle AFK : \triangle AFB.$$

$$\therefore \triangle ALN \Bumpeq \triangle AFB \Bumpeq \text{square } BD.$$

Ex. 474. To construct an equilateral triangle having a given area.

The construction is effected as in Ex. 473, each $\angle$ being 60°.

Ex. 475. To transform the triangle ABC into a triangle similar to a given triangle PQR.



Analysis. Let AFG be the required △, constructed with the side AF in the line AB. Draw $CD \parallel$ to AB and meeting AG in D, and $DE \parallel$ to GF; then, by similar ▲,

$$\triangle AED : \triangle AFG = \overline{AE}^2 : \overline{AF}^2;$$

or, since

$$\triangle AFG \approx \triangle ABC,$$

$$\triangle AED : \triangle ABC = \overline{AE}^2 : \overline{AF}^2.$$

Since the ▲ AED and ABC have equal altitudes,

$$\triangle AED : \triangle ABC = AE : AB.$$

$$\therefore \overline{AE}^2 : \overline{AF}^2 = AE : AB,$$

and

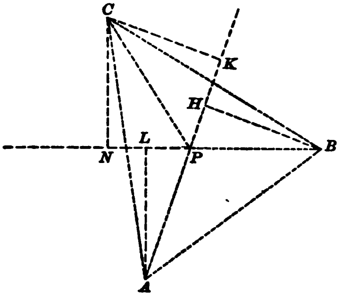
$$\overline{AF}^2 = AB \times AE.$$

That is, AF is the mean proportional between the two known lines AB and AE.

Ex. 476. Given three points A, B, C. To find a fourth point P such that the areas of the triangles APB, APC, BPC shall be equal.

Let A, B, C be the three given points. It is required to find a fourth point P, so that the

$$\triangle APB \approx \triangle APC \approx \triangle BPC.$$



Analysis. Suppose the problem solved, and P the required fourth point. The ▲ BPC and APB are equivalent, and have a common base PB; hence their altitudes CN and AL are equal (§ 370). In like manner, since the ▲ APB and APC are equivalent, and have a common base AP, their altitudes BH and CK are equal. But since $CN = AL$, AC must be bisected by BP (§ 149); and in like manner, since $CK = BH$, BC must be bisected by AP.

CONSTRUCTION. Join the three points A, B, C. Draw BP and AK through the middle points of AC and BC respectively, and intersecting each other in P; then will P be the fourth point required.

PROOF. Since BC is bisected by AP (Cons.), the two rt. ▲ formed by the halves of BC and the $\perp$s BH and CK are equal (§ 148); hence $BH = CK$.

$$\therefore \triangle APB \Leftrightarrow \triangle APC. \qquad (§\ 369)$$

Also, since AC is bisected by BP (Cons.), the two rt. ▲ formed by the halves of AC and the $\perp$s CN and AL are equal (§ 148); hence $CN = AL$.

$$\therefore \triangle APB \Leftrightarrow \triangle BPC. \qquad (§\ 369)$$

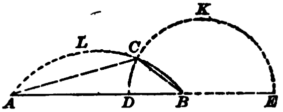
$$\text{But } \triangle APB \Leftrightarrow \triangle APC. \quad \therefore \triangle APB \Leftrightarrow \triangle APC \Leftrightarrow \triangle BPC.$$

Ex. 477. To construct a triangle, given its base, the ratio of the other sides, and the angle included by them.

Let AB be the given base, $m : n$ the given ratio, and ACB the given $\angle$. It is required to construct a $\triangle ABC$

so that $CA : CB = m : n$,

and $\angle ACB =$ the given $\angle$.



ANALYSIS. Since $CA : CB = m : n$, if AB be divided harmonically at D and E, in the given ratio, C must lie on the circumference described on DE as diameter (Ex. 458).

Since $\angle ACB =$ the given $\angle$, C must be on the arc ALB, which contains the given $\angle$ (§ 290). Hence the point C is determined by the intersection of arcs ALB and DKE.

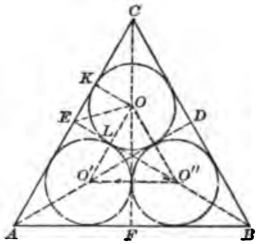
CONSTRUCTION. Divide AB harmonically at D and E. Upon DE as a diameter describe semicircle DKE. Upon AB describe segment ALB containing the given $\angle$ (§ 290). Let arc ALB cut DKE at C. Join AC and CB. Then will ACB be the required $\triangle$.

PROOF. $\angle ACB$ is required $\angle$, and $CA : CB = m : n$. (Ex. 458)

Ex. 478. To divide a given circle into any number of equivalent parts by concentric circumferences.

SOLUTION. The method employed in Ex. 437 is applicable to the division of a $\odot$ into any number of equal parts, the number of equal parts into which the radius is divided corresponding to the required number of parts of the $\odot$.

Ex. 479. In a given equilateral triangle to inscribe three equal circles tangent to each other and each circle tangent to two sides of the triangle.



Let *ABC* be an equilateral △. It is required to draw three equal Ⓢ, each tangent to the other two and to two sides of the △.

Analysis. Each of these Ⓢ is evidently inscribed in a quadrilateral formed by the sides and altitudes of the △, as *ECDS* (*S*, not shown in the figure, being the intersection of the altitudes). The centre of the ⊙ must lie on *CF*, as *CF* is the locus of all points equidistant from *CA* and *CB*. As this ⊙ must also touch *BE*, its centre lies on *EO*, the bisector of the ∠ *CEB*. Hence the centre is at *O* where *CF* and *EO* intersect, and the radius *OK*, the ⊥ from *O* upon *CE*, *SE*, *SD*, or *CD*.

Construction. Draw the altitudes *CF*, *BE*, *AD*. Bisect the ∠ *CEB* by *EO*, intersecting *CF* in *O*. From *O* draw the ⊥ *OK* to *CA*. Then with *O* as a centre and *OK* as a radius describe a ⊙. This ⊙ will be inscribed in *ECDS* and will be one of the required Ⓢ. The same construction may be repeated for each of the other quadrilaterals *EAFS* and *FBDS*.

Proof. Since *CF* bisects the ∠ *ACB*, *O* is equidistant from *CA* and *AB* (§ 162). Since *EO* bisects the ∠ *CEB*, *O* is equidistant from *EC* and *ES*. Since *ECDS* is symmetrical with respect to *CS* (§ 208), *O* is equidistant from *ES* and *DS*.

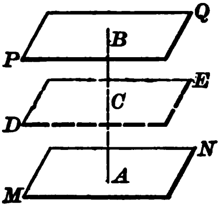
Discussion. Since *O*, *O'*, *O''* are equidistant from the sides, the centres *O'*, *O''* may also be found by drawing *OO'* ∥ to *CA*, and *O'O''* ∥ to *AB*. The intersections of these lines with *AD* and *BE* determine *O'* and *O''*.

SOLID GEOMETRY.

BOOK VI.

Ex. 480. Find the locus of a point in space equidistant from two given parallel planes.

Let *MN* and *PQ* be any two ∥ planes. It is required to find the locus of a point equidistant from these planes.



CONSTRUCTION. At any point *A* in *MN* erect a ⊥, and produce it to meet *PQ* in some point, as *B*. Bisect *AB*, and through its middle point *C* pass a plane *DE* ⊥ to *AB*; then will the plane *DE* be the locus required.

PROOF. The plane *DE* is ∥ to *MN* (§ 491).

Hence the planes *MN*, *DE*, and *PQ* are all parallel.

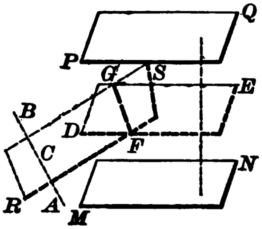
Any point in *DE* is at the distance *AC* from the plane *MN*.

For like reason any point in *DE* is at the distance *BC* from the plane *PQ*.

But, by construction, $AC = BC$; hence any point in the plane *DE* is equally distant from *MN* and *PQ*.

Any point without the plane *DE* is evidently nearer one plane than the other; hence the plane *DE* is the required locus.

Ex. 481. Find the locus of a point in space equidistant from two given points, and also equidistant from two given parallel planes.

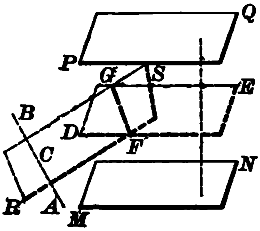


Let *A* and *B* be any two given points in space, and *MN* and *PQ* any two given ∥ planes. It is required to find the locus of a point equidistant from *A* and *B*, and also equidistant from the planes *MN* and *PQ*.

CONSTRUCTION. Draw *AB*, bisect it;

through C, its middle point, pass a plane $RS \perp$ to AB. Construct the plane $DE \parallel$ to MN and PQ, and midway between them.

FG, the intersection of the planes RS and DE, will be the locus required.



PROOF. Any point in the plane RS is equally distant from A and B (§ 482).

Any point in the plane DE is equally distant from the planes MN and PQ, by Ex. 480. Hence any point in FG, the intersection of the planes RS and DE, is equally distant from the points A and B, and also from the planes MN and PQ, and therefore FG is the required locus.

DISCUSSION. If the line AB is $\perp$ to the plane MN, there will be no locus unless the middle point of AB is midway between the planes MN and PQ, in which case the locus will be the plane DE. In all other cases the locus will be a straight line.

Ex. 482. The line AB meets three parallel planes in the points A, E, B; and the line CD meets the same planes in the points C, F, D. If $AE = 6$ inches, $BE = 8$ inches, $CD = 12$ inches, compute CF and FD.

$$AE : EB = CF : FD. \quad (\S\ 499)$$

$$\therefore AE + EB : EB = CF + FD : FD,$$

or

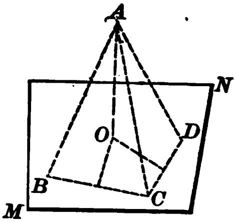
$$AB : EB = CD : FD.$$

$$14 : 8 = 12 : FD.$$

$$FD = \frac{8 \times 12}{14} = \frac{48}{7} = 6\tfrac{6}{7} \text{ inches.}$$

$$CF = CD - FD = 12 - 6\tfrac{6}{7} = 5\tfrac{1}{7} \text{ inches.}$$

Ex. 483. To draw a perpendicular to a given plane from a given point without it.

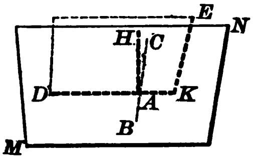


Let MN be any given plane, and A any given point without it. It is required to draw a $\perp$ from A to the plane MN.

CONSTRUCTION. From A draw any three equal oblique lines to the plane MN, as AB, AC, and AD. In the plane MN find the point O equally distant from the points B, C, and D. Draw AO, which will be the $\perp$ required.

PROOF. The foot of the ⊥ from A to the plane MN is equally distant from the feet of the equal oblique lines AB, AC, and AD (§ 479). But O is the only point in MN that is equally distant from B, C, and D; hence AO is the ⊥ from A to MN.

Ex. 484. To draw a perpendicular to a given plane at a given point in it.



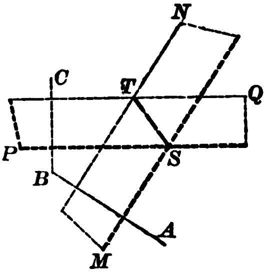
Let A be any point in the given plane MN. It is required to draw a ⊥ to the plane MN at the point A.

CONSTRUCTION. Through A draw any line BC in the plane MN. Through A draw the plane ED ⊥ to BC, cutting MN in DAK. In the plane ED draw AH ⊥ to DK; then will HA be the ⊥ required.

PROOF. Since BC is ⊥ to the plane ED, it is ⊥ to HA in that plane. But HA, by construction, is ⊥ to DK. Hence HA is ⊥ to two lines BC and DK of the plane MN at their intersection A, and is therefore ⊥ to the plane MN at A (§ 472).

Ex. 485. Find the locus of a point in space equidistant from three given points not in a straight line.

Let A, B, and C be any three points in space. It is required to find the locus of a point in space equidistant from A, B, and C.

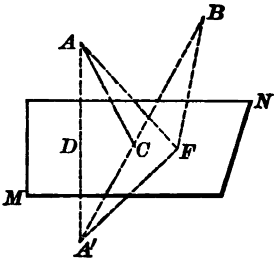


CONSTRUCTION. Draw AB and BC. At the middle point of AB draw the plane MN ⊥ to AB, and at the middle point of BC draw the plane PQ ⊥ to BC. Let ST be the intersection of the planes MN and PQ; then will ST be the locus required.

PROOF. Since ST is in the plane MN, any point in ST is equally distant from A and B (§ 482). For like reason any point in ST is equally distant from B and C. Hence any point in ST is equally distant from A, B, and C. Any point without this line is without one or both of the planes MN and PQ, and is therefore not equally distant from the points A, B, and C.

Hence ST is the locus required.

Ex. 486. Given two points A and B on the same side of a given plane MN; find a point in this plane such that the sum of its distances from A and B shall be a minimum.



Construction. From A draw $AD \perp$ to the plane MN, and produce it, making $DA' = AD$. Draw $A'B$, cutting MN in C. Draw AC; then will $AC + BC$ be less than the sum of any other two lines drawn from A and B to any point in MN.

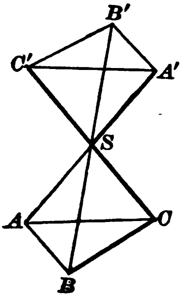
Proof. Let F be any other point than C of the plane MN. Draw FB, FA, and FA'.

Now $$FA = FA',\ CA = CA' \text{ (§ 482)}, \qquad (1)$$

$$BF + FA' > BC + CA' \text{ (§ 33)}, \qquad (2)$$

Substituting for FA' and CA' in (2) their equals given in (1), we obtain $$BF + FA > BC + CA.$$

Ex. 487. An isosceles trihedral angle and its symmetrical trihedral angle are superposable.



Let $S\text{-}ABC$ be an isosceles trihedral $\angle$, and $S\text{-}A'B'C'$ its symmetrical trihedral $\angle$. We are to prove that they are superposable.

Proof. $\angle ASB = \angle BSC$. (Hyp.)

$\angle C'SB' = \angle BSC$. (§ 95)

$\therefore \angle ASB = \angle C'SB'$.

For like reason $\angle BSC = \angle B'SA'$.

Dihedral $A\text{-}BS\text{-}C =$ dihedral $C'\text{-}B'S\text{-}A'$,

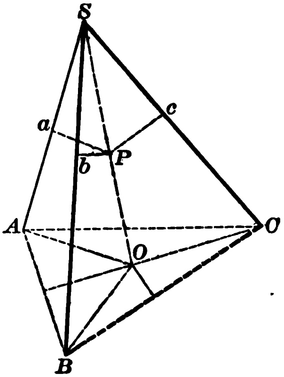
(opposite dihedrals are equal).

Revolve $S\text{-}A'B'C'$ about S until SB' falls on SB, and the equal dihedrals SB' and SB coincide.

Then, since $\angle B'SA' = \angle BSC$, SA' will coincide with SC. For like reason, SC' will fall on SA. Hence the edges of the trihedrals, and therefore the trihedrals themselves, coincide.

Ex. 488. Find the locus of a point equidistant from the three edges of a trihedral angle.

Let S-ABC be any trihedral ∠. It is required to find the locus of a point equidistant from the three edges SA, SB, and SC.



Construction. Lay off $SA = SB = SC$ and draw the lines AB, BC, and AC.

In the plane ABC find the point O equally distant from A, B, and C. Draw SO; then will SO be the locus required.

Proof. Since O is equally distant from A, B, and C, and $SA = SB = SC$, and OS is common, the ▲ OSA, OSB, and OSC are mutually equilateral, and equal.

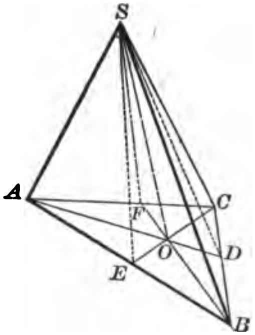
Hence the ∠s OSA, OSB, and OSC are equal.

From P, any point in SO, draw the ⊥s Pa, Pb, and Pc to the edges SA, SB, and SC, respectively. The rt. ▲ SaP, SbP, and ScP are equal, since the side SP is common and the ∠s at S are equal (§ 148). Whence $Pa = Pb = Pc$, or the point P is equally distant from the three edges SA, SB, and SC.

The line drawn from S to any point without SO would not pierce the plane ABC at a point equidistant from A, B, and C, and therefore would not make equal ∠s with the edges (§ 153); hence the ⊥s from this point to the edges would not be equal.

Therefore SO is the locus of all points equidistant from the edges SA, SB, and SC.

Ex. 489. Find the locus of a point equidistant from the three faces of a trihedral angle.



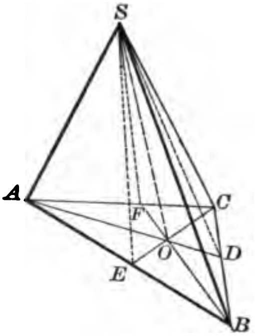
Let S-ABC be any trihedral ∠. It is required to find the locus of a point equidistant from the three faces SAB, SBC, and SAC.

Construction. Draw the plane SBF, bisecting the dihedral ∠ SB; also the plane SAD, bisecting the dihedral ∠ SA. Let SO be the intersection of these bisecting planes; then is SO the locus required.

Proof. Since any point in SO is in the plane SBF, it is equally distant from the faces SBA and SBC (§ 525).

For like reason any point in *SO* is equally distant from the faces *SAB* and *SAC.*

Hence any point in *SO* is equally distant from the faces *SAB*, *SBC*, and *SAC.*



Any point without *SO* is without one or both of the planes *SBF* and *SAD*, and is therefore not equally distant from the faces *SAB*, *SBC*, and *SAC.*

Hence *SO* is the locus required.

Book VII.

Ex. 490. Show that any lateral edge of a right prism is equal to the altitude.

Proof. Any lateral edge of a right prism is ⊥ to the planes of its bases, and therefore measures the distance between these bases; that is, it equals the altitude of the prism.

Ex. 491. Show that the lateral faces of right prisms are rectangles.

Proof. The lateral edges of any right prism are ⊥ to the planes of its bases. Hence these lateral edges are ⊥ to the sides of the bases of the prism, or the ∠s of the lateral faces are rt. ∠s; that is, these faces are rectangles.

Ex. 492. Prove that every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

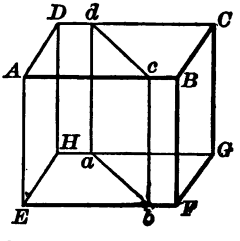
Let *abcd* be a section of the prism *AG*, made by the plane *abc* ∥ to the lateral edges *AE*, *BF*, *CG*, and *DH*.

We are to prove that *abcd* is a ▱.

Proof. The lines *ab* and *dc* are parallel (§ 492).

Since the plane *abc* and the edge *BF* are ∥, the lines *bc* and *FB* cannot meet, however far they are produced; hence, as they lie in the same plane *ABF*, these lines are parallel.

For like reason *ad* and *HD* are parallel.



But *DH* and *BF* are parallel; hence *bc* and *ad* are parallel. (§ 111)

Therefore *abcd*, having its opposite sides ∥, is a ▱.

Ex. 493. Find the ratio of two rectangular parallelopipeds if their altitudes are each 6 inches, and their bases 5 inches by 4 inches, and 10 inches by 8 inches, respectively.

$$\frac{P}{P'}=\frac{\cancel{6}\times\cancel{5}\times\cancel{4}}{\cancel{6}\times\underset{2}{\cancel{10}}\times\underset{2}{\cancel{8}}}=\frac{1}{4}$$

Ex. 494. Find the ratio of two rectangular parallelopipeds, if their dimensions are 3, 4, 5, and 9, 8, 10, respectively.

$$\frac{P}{P'}=\frac{\cancel{3}\times\cancel{4}\times\cancel{5}}{\underset{3}{\cancel{9}}\times\underset{2}{\cancel{8}}\times\underset{2}{\cancel{10}}}=\frac{1}{12}.$$

Ex. 495. Find the volume of a right triangular prism, if its height is 14 inches, and the sides of the base are 6, 5, and 5 inches.

$$\text{Altitude of base} = \sqrt{5^2-3^2}=\sqrt{25-9}=\sqrt{16}=4 \text{ in.}$$

$$\text{Area of base} = \tfrac{1}{2}\times 4\times 6 \text{ sq. in.}$$

$$V=14\times\tfrac{1}{2}\times 4\times 6=168 \text{ cu. in.}$$

Ex. 496. Find the length of an edge of a cubical vessel which will hold 2 tons of water.

$$2 \text{ t.} = 2\times 2000 \text{ lbs.} = 4000 \text{ lbs.}$$

$$4000 \text{ lbs.} \div 62\tfrac{1}{2} \text{ lbs.} = \frac{2\times 4000}{125} \text{ cu. ft.}$$

$$\text{The number of feet on a side} = \sqrt[3]{\tfrac{8000}{125}}=\sqrt[3]{64}=4 \text{ ft.}$$

Ex. 497. How many square feet of lead will be required to line a cistern, open at the top, which is 4 feet 6 inches long, 2 feet 8 inches wide, and contains 42 cubic feet?

$$4 \text{ ft. } 6 \text{ in.} = 4\tfrac{1}{2} \text{ ft.}$$

$$2 \text{ ft. } 8 \text{ in.} = 2\tfrac{2}{3} \text{ ft.}$$

$$42\div(4\tfrac{1}{2}\times 2\tfrac{2}{3})=\overset{\overset{7}{\cancel{14}}}{\cancel{42}}\times\frac{\cancel{2}}{\underset{\cancel{3}}{\cancel{9}}}\times\frac{\cancel{3}}{\underset{2}{\cancel{8}}}=\frac{7}{2}=3\tfrac{1}{2} \text{ ft. high.}$$

$$4\tfrac{1}{2}+2\tfrac{2}{3}=7\tfrac{1}{6}.$$

$$7\tfrac{1}{6}\times 2\times 3\tfrac{1}{2}=\frac{43}{6}\times\cancel{2}\times\frac{7}{\cancel{2}}=\frac{301}{6}=50\tfrac{1}{6} \text{ sq. ft., sides.}$$

$$4\tfrac{1}{2}\times 2\tfrac{2}{3}=\frac{\overset{3}{\cancel{9}}}{\cancel{2}}\times\frac{\overset{4}{\cancel{8}}}{\cancel{3}}=12 \text{ sq. ft., bottom.}$$

$$50\tfrac{1}{6}+12=62\tfrac{1}{6} \text{ sq. ft.} = 62 \text{ sq. ft. } 24 \text{ sq. in.}$$

Ex. 498. An open cistern is made of iron 2 inches thick. The inner dimensions are: length, 4 feet 6 inches; breadth, 3 feet; depth, 2 feet

6 inches. What will the cistern weigh (i.) when empty? (ii.) when full of water? Specific gravity of iron = 7.2.

$$4 \text{ ft. } 6 \text{ in.} + 4 \text{ in.} = 4 \text{ ft. } 10 \text{ in.} = 4\tfrac{5}{6} \text{ ft.}$$

$$3 \text{ ft.} + 4 \text{ in.} = 3 \text{ ft. } 4 \text{ in.} = 3\tfrac{1}{3} \text{ ft.}$$

$$4\tfrac{5}{6} \times 3\tfrac{1}{3} \times \tfrac{1}{6} = \frac{29}{6} \times \frac{\overset{5}{\cancel{10}}}{3} \times \frac{1}{\underset{3}{\cancel{6}}} = \frac{145}{54} = 2\tfrac{37}{54} \text{ cu. ft., bottom.}$$

$$4\tfrac{5}{6} \times 2\tfrac{1}{2} \times \tfrac{1}{6} \times 2 = \frac{29}{6} \times \frac{5}{\cancel{2}} \times \frac{1}{6} \times \cancel{2} = \frac{145}{36} = 4\tfrac{1}{36} \text{ cu. ft., sides.}$$

$$3 \times 2\tfrac{1}{2} \times \tfrac{1}{6} \times 2 = \cancel{3} \times \frac{5}{2} \times \frac{1}{\cancel{6}} \times \cancel{2} = \frac{5}{2} = 2\tfrac{1}{2} \text{ cu. ft., ends.}$$

$$2\tfrac{37}{54} + 4\tfrac{1}{36} + 2\tfrac{1}{2} = 9\tfrac{23}{108} \text{ cu. ft.}$$

$$9\tfrac{23}{108} \times 62\tfrac{1}{2} \times 7\tfrac{1}{5} = \frac{995}{\underset{3}{\cancel{108}}} \times \frac{\overset{25}{\cancel{125}}}{2} \times \frac{\cancel{36}}{\cancel{5}} = \frac{24875}{6} = 4145\tfrac{5}{6} \text{ lbs.}$$

$$= 4145 \text{ lbs. } 13 \text{ oz. empty.}$$

$$4\tfrac{1}{2} \times 3 \times 2\tfrac{1}{2} \times 62\tfrac{1}{2} = \frac{9}{2} \times 3 \times \frac{3}{2} \times \frac{125}{2} = \frac{16875}{8} = 2109\tfrac{3}{8} \text{ lbs.}$$

$$= 2109 \text{ lbs. } 6 \text{ oz.}$$

$$4145 \text{ lbs. } 13 \text{ oz.} + 2109 \text{ lbs. } 6 \text{ oz.} = 6255 \text{ lbs. } 3 \text{ oz. full.}$$

Ex. 499. An open cistern 6 feet long and 4½ feet wide holds 108 cubic feet of water. How many cubic feet of lead will it take to line the sides and bottom, if the lead is ⅛ inch thick?

$$\cancel{6} \times 4\tfrac{1}{2} = 27 \text{ sq. ft., bottom.}$$

$$\text{Height} = 108 + 27 = 4 \text{ ft.}$$

$$4 \times 2(6 + 4\tfrac{1}{2}) = 4 \times 2 \times \tfrac{21}{2} = 84 \text{ sq. ft., sides.}$$

$$27 + 84 = 111 \text{ sq. ft.}$$

$$\overset{37}{\cancel{111}} \times \frac{1}{8 \times \underset{4.}{\cancel{12}}} = \frac{37}{32} = 1\tfrac{5}{32} \text{ cu. ft.} = 1 \text{ cu. ft. } 270 \text{ cu. in.}$$

Ex. 500. The three dimensions of a rectangular parallelopiped are a, b, c; find the surface, the volume, and the length of a diagonal.

$$S = 2ab + 2ac + 2bc = 2(ab + ac + bc).$$

$$V = abc.$$

$$\text{Diagonal of base} = \sqrt{a^2 + b^2}.$$

$$\therefore \text{Diagonal} = \sqrt{(\sqrt{a^2 + b^2})^2 + c^2} = \sqrt{a^2 + b^2 + c^2}.$$

Ex. 501. The base of a right prism is a rhombus, one side of which is 10 inches, and the shorter diagonal is 12 inches. The height of the prism is 15 inches. Find the entire surface and the volume.

$$\tfrac{1}{2} \text{ longer diagonal of base} = \sqrt{10^2 - 6^2} = 8.$$

$$\begin{aligned}
\textbf{Base} &= 4(\tfrac{1}{2} \times 6 \times 8) = 96. \\
\text{Sides} &= 4 \times 10 \times 15 = 600 \text{ sq. in.} \\
\text{Bases} &= 2 \times 96 = 192 \text{ sq. in.} \\
T &= 600 + 192 = 792 \text{ sq. in.} \\
V &= 15 \times 96 = 1440 \text{ cu. in.}
\end{aligned}$$

Ex. 502. Find the volume of a right hexagonal prism whose height is 10 feet, each side of the hexagon being 10 inches.

$$a = \sqrt{10^2 - 5^2} = \sqrt{100 - 25} = \sqrt{75} = 8.66025.$$

$$\begin{aligned}
\text{Base} &= 6 \times \tfrac{1}{2}(10 \times 8.66025) = 259.8075 \text{ sq. in.} \\
\text{Altitude} &= 10 \text{ ft.} = 120 \text{ in.} \\
V &= 120 \times 259.8075 = 31176.9 \text{ cu. in.} \\
&= 18 \text{ cu. ft. } 73 \text{ cu. in.}
\end{aligned}$$

Ex. 503. A pyramid 15 feet high has a base containing 169 square feet. At what distance from the vertex must a plane be passed parallel to the base so that the section may contain 100 square feet?

$$\begin{aligned}
\sqrt{169} : \sqrt{100} &= 15 : x, \\
13 : 10 &= 15 : x, \\
13x &= 150, \\
x &= 11\tfrac{7}{13} = 11.54 \text{ ft.}
\end{aligned}$$

Ex. 504. The base of a pyramid contains 144 square feet. A plane parallel to the base and 4 feet from the vertex cuts a section containing 64 square feet; find the height of the pyramid.

$$\begin{aligned}
\sqrt{64} : \sqrt{144} &= 4 : h, \\
8 : 12 &= 4 : h, \\
2 : 3 &= 4 : h, \\
2h &= 12, \\
h &= 6 \text{ ft.}
\end{aligned}$$

Ex. 505. A pyramid 12 feet high has a square base measuring 8 feet on a side. What will be the area of a section made by a plane parallel to the base and 4 feet from the vertex?

$$\begin{aligned} 12^2 : 4^2 &= 8^2 : x, \\ 144 : 16 &= 64 : x, \\ 9 : 1 &= 64 : x, \\ x &= \tfrac{64}{9} = 7.11 \text{ sq. ft.} \end{aligned}$$

Ex. 506. Two pyramids standing on the same plane are 14 feet high. The first has for base a square measuring 9 feet on a side; the second a hexagon measuring 7 feet on a side. Find the areas of the sections made by a plane parallel to their bases and 6 feet from their vertices.

Let a = the apothem of the base.

$$14^2 : 6^2 = 9^2 : x,$$

$$49 : 9 = 81 : x,$$

$$x = \frac{9 \times 81}{49} = \frac{729}{49} = 14.88 \text{ sq. ft.}$$

$$a = \sqrt{7^2 - 3.5^2} = \sqrt{49 - 12.25} = \sqrt{36.75} = 6.0622.$$

$$\text{Base} = 6 \times \tfrac{1}{2} \times 7 \times 6.0622 = 21 \times 6.0622 = 127.3062 \text{ sq. ft.}$$

$$14^2 : 6^2 = 127.3062 : x$$

$$49 : 9 = 127.3062 : x,$$

$$x = \frac{9 \times 127.3062}{49} = \frac{1145.7558}{49} = 23.38 \text{ sq. ft.}$$

Ex. 507. The base of a regular pyramid is a hexagon of which the side measures 3 feet. Find the height of the pyramid if the lateral area is equal to ten times the area of the base.

$$a = \sqrt{3^2 - 1.5^2} = \sqrt{9 - 2.25} = \sqrt{6.75}.$$

$$\begin{aligned} x &= \sqrt{(10\sqrt{6.75})^2 - (\sqrt{6.75})^2} = \sqrt{675 - 6.75} \\ &= \sqrt{668.25} = 25.85 \text{ ft.} \end{aligned}$$

Ex. 508. The homologous edges of two similar tetrahedrons are as 6 : 7. Find the ratio of their surfaces and of their volumes.

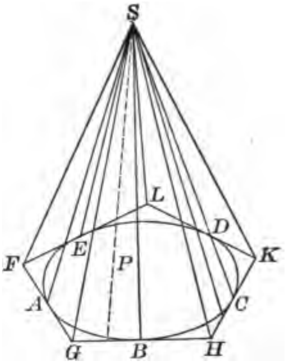
$$S : S' = 6^2 : 7^2 = 36 : 49.$$

$$V : V' = 6^3 : 7^3 = 216 : 343.$$

Ex. 509. If the edge of a tetrahedron is a, find the homologous edge of a similar tetrahedron twice as large.

$$V : V' = a^3 : 2a^3. \quad \text{Edge of } V' = a\sqrt[3]{2}.$$

Ex. 510. Show that any lateral face of a pyramid circumscribed about a cone is tangent to the cone.



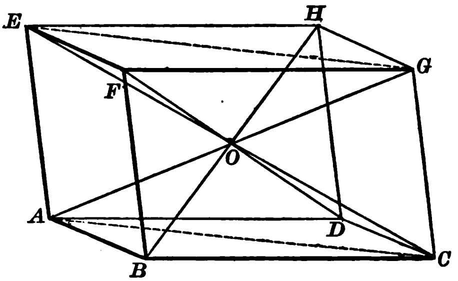
Let *S-FGHKL* be a pyramid circumscribed about the cone *S-ABCDE*. We are to prove that any face, as *GSH*, is tangent to the cone.

Proof. Let *A*, *B*, *C*, *D*, *E* be the points of contact of the sides of the base of the pyramid with the base of the cone; then are *SA*, *SB*, *SC*, *SD*, *SE*, elements of the surface of the cone.

Any point *P* in face *GSH* without the line *SB* is without the cone. For the line *SP* produced will meet the plane of the base of the cone in some point of *GH* other than *B*; that is, in some point without the base of the cone. But every point, other than *S*, in such a line lies without the surface of the cone.

Hence, by definition, the face *GSH* is tangent to the surface of the cone, *SB* being the element of contact. In like manner it may be shown that any other lateral face of the pyramid is tangent to the surface of the cone.

Ex. 511. The diagonals of a parallelopiped bisect each other.



Let *AG*, *EC*, *BH*, and *FD* be the four diagonals of the parallelopiped *AG*.

We are to prove that these four diagonals bisect each other.

Proof. Through the opposite and ∥ edges *AE* and *CG* pass a plane intersecting the ∥ bases in the ∥ lines *AC* and *EG*. The section *ACGE* is a ▱, having its opposite sides ∥.

$\therefore$ its diagonals *AG* and *EC* bisect each other. (§ 184)

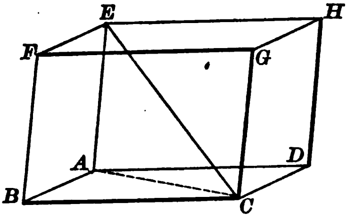
That is, *O* is the middle point of *AG* and *EC*.

In like manner a plane passed through the opposite and $\parallel$ edges *FG* and *AD* will form a ▱ *AFGD*; hence *O*, the middle point of the diagonal *AG*, is also the middle point of *FD* (§ 184).

Also, a plane passed through the opposite and $\parallel$ edges *EH* and *BC* will form a ▱ *EBCH*; hence *O*, the middle point of the diagonal *EC*, is also the middle point of *BH* (§ 184).

Hence the four diagonals bisect each other at the point *O*.

Ex. 512. The square of the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three dimensions.



Let *CE* be a diagonal, and *CD*, *CB*, and *CG* the dimensions of the rectangular parallelopiped *E-ABCD*: we are to prove that

$$\overline{CE}^2 = \overline{CD}^2 + \overline{CB}^2 + \overline{CG}^2.$$

PROOF. Draw *CA*; then the $\triangle$ *ACD* and *ACE* are rt. $\triangle$.

Hence $\overline{CA}^2 = \overline{CD}^2 + \overline{AD}^2 = \overline{CD}^2 + \overline{CB}^2,$ (§ 338)

and $\overline{CE}^2 = \overline{CA}^2 + \overline{AE}^2 = \overline{CA}^2 + \overline{CG}^2.$ (§ 338)

$$\therefore \overline{CE}^2 = \overline{CD}^2 + \overline{CB}^2 + \overline{CG}^2.$$

Ex. 513. Find the volume in cubic feet of a regular pyramid when its base is a square, each side measuring 3 feet 4 inches, and its height is 9 feet.

3 ft. 4 in. = $3\frac{1}{3}$ feet.

$$V = \tfrac{1}{3} \times 3\tfrac{1}{3} \times 3\tfrac{1}{3} \times 9 = \frac{1}{3} \times \frac{10}{\cancel{3}} \times \frac{10}{\cancel{3}} \times \overset{\cancel{3}}{\cancel{9}} = \frac{100}{3} = 33\tfrac{1}{3} \text{ cu. ft.}$$

Ex. 514. Find the volume in cubic feet of a regular pyramid when its base is an equilateral triangle, each side measuring 4 feet, and its height is 15 feet.

$$a = \sqrt{4^2 - 2^2} = \sqrt{12} = 3.464.$$

$$V = \tfrac{1}{3} \times 15(\tfrac{1}{2} \times 4 \times 3.464) = 10 \times 3.464 = 34.64 \text{ cu. ft.}$$

Ex. 515. Find the volume in cubic feet of a regular pyramid when its base is a regular hexagon, each side measuring 6 feet, and its height is 30 feet.

$$a = \sqrt{6^2 - 3^2} = \sqrt{36 - 9} = \sqrt{27} = 5.1961.$$

$$\text{Base} = 6 \times \tfrac{1}{2} \times 6 \times 5.1961 = 18 \times 5.1961.$$

$$V = \frac{1}{\cancel{3}} \times \overset{10}{\cancel{30}} \times 18 \times 5.1961 = 180 \times 5.1961 = 935.298 \text{ cu. ft.}$$
$$= 935.3 \text{ cu. ft.}$$

Ex. 516. Find the total surface in square feet of a regular pyramid when each side of its square base is 8 feet, and the slant height is 20 feet.

$$T = 4 \times \tfrac{1}{2}(20 \times 8) + 8^2 = 320 + 64 = 384 \text{ sq. ft.}$$

Ex. 517. Find the total surface in square feet of a regular pyramid when each side of its triangular base is 6 feet, and the slant height is 18 feet.

$$\text{Altitude of base} = \sqrt{6^2 - 3^2} = 5.196.$$

$$T = 3 \times \tfrac{1}{2}(6 \times 18) + \tfrac{1}{2} \times 6 \times 5.196 = 162 + 15.588$$
$$= 177.588 \text{ sq. ft.}$$

Ex. 518. Find the total surface in square feet of a regular pyramid when each side of its square base is 26 feet, and the perpendicular height is 84 feet.

$$a = \sqrt{84^2 + 13^2} = \sqrt{7056 + 169} = \sqrt{7225} = 85.$$

$$T = 4 \times \tfrac{1}{2}(26 \times 85) + 26^2 = 4420 + 676 = 5096 \text{ sq. ft.}$$

Ex. 519. Find the height in feet of a regular pyramid when the volume is 26 cubic feet 936 cubic inches, and each side of its square base is 3 feet 6 inches.

$$26 \text{ cu. ft. } 936 \text{ cu. in.} = 26\tfrac{13}{24} \text{ cu. ft.}$$

$$3 \text{ ft. } 6 \text{ in.} = 3\tfrac{1}{2} \text{ ft.}$$

$$V = \tfrac{1}{3} H \times 3\tfrac{1}{2} \times 3\tfrac{1}{2}.$$

$$26\tfrac{13}{24} = \tfrac{1}{3} H \times \tfrac{7}{2} \times \tfrac{7}{2}.$$

$$H = \frac{26\frac{13}{24}}{\frac{1}{3} \times \frac{7}{2} \times \frac{7}{2}} = \frac{\overset{\overset{13}{\cancel{91}}}{\cancel{637}}}{\underset{\underset{\underset{2}{\cancel{4}}}{\cancel{8}}}{\cancel{24}}} \times \frac{\cancel{3}}{1} \times \frac{\cancel{2}}{\cancel{7}} \times \frac{\cancel{2}}{\cancel{7}} = \frac{13}{2} = 6\tfrac{1}{2} \text{ ft.}$$

Ex. 520. Find the height in feet of a pyramid when the volume is 20 cubic feet, and the sides of its triangular base are 5 feet, 4 feet, and 3 feet.

$$\text{Base} = 6 \text{ sq. ft.}$$
$$V = \tfrac{1}{3} B \times h.$$
$$20 = \tfrac{1}{3} \times 6h.$$
$$h = \frac{20}{\tfrac{1}{3} \times 6} = 10 \text{ ft.}$$

Ex. 521. The base edge of a regular pyramid with a square base measures 40 feet, the lateral edge 101 feet; find its volume in cubic feet.

$\sqrt{40^2 + 40^2} = 40\sqrt{2}$, diagonal of base.

$\tfrac{1}{2}$ of $40\sqrt{2} = 20\sqrt{2}$ ft.

$\sqrt{101^2 - (20\sqrt{2})^2} = \sqrt{10201 - 800} = \sqrt{9401} = 96.9587$ ft.

$V = \tfrac{1}{3} \times 40^2 \times 96.9587 = 51711.31$ cu. ft.

Ex. 522. Find the volume of a regular pyramid whose slant height is 12 feet, and whose base is an equilateral triangle inscribed in a circle having a radius of 10 feet.

Let $2x$ denote a side of the triangle.

The altitude of the triangular base is 15 ft. (Ex. 33)

$$4x^2 = x^2 + 225,$$
$$3x^2 = 225,$$
$$x^2 = 75,$$
$$x = 8.660,$$
$$2x = 17.320.$$
$$h = \sqrt{12^2 - 5^2} = \sqrt{144 - 25} = \sqrt{119} = 10.908.$$
$$V = \tfrac{1}{3} \times \tfrac{1}{2}(15 \times 17.320) \times 10.908 = 472.3164 \text{ cu. ft.}$$

Ex. 523. Having given the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.

Let x denote the slant height.

Then
$$x = \sqrt{\tfrac{1}{4}a^2 + h^2}.$$
$$T = 2ax + a^2,$$
$$T = 2a\sqrt{\tfrac{1}{4}a^2 + h^2} + a^2.$$
$$\sqrt{\tfrac{1}{4}a^2 + h^2} = \frac{T - a^2}{2a}.$$
$$\tfrac{1}{4}a^2 + h^2 = \frac{T^2 - 2a^2T + a^4}{4a^2}.$$
$$h^2 = \frac{T^2 - 2a^2T}{4a^2},$$

$$h=\frac{1}{2a}\sqrt{T(T-2a^2)}.$$

$$V=\tfrac{1}{3}a^2h=\tfrac{1}{3}a^2\times\frac{1}{2a}\sqrt{T(T-2a^2)}=\frac{a}{6}\sqrt{T(T-2a^2)}.$$

Ex. 524. The base edge of a regular pyramid whose base is a square is a, the total surface T; find the height of the pyramid.

$$h=\frac{1}{2a}\sqrt{T(T-2a^2)}. \qquad \text{(Ex. 523.)}$$

Ex. 525. The eight edges of a regular pyramid with a square base are equal in length, and the total surface is T; find the length of one edge.

Let x = slant height.

$$x=\sqrt{a^2-\frac{a^2}{4}}=\sqrt{\frac{3a^2}{4}}=\frac{a}{2}\sqrt{3}.$$

$$T=2ax+a^2,$$

$$T=a^2\sqrt{3}+a^2.$$

$$a^2=\frac{T}{1+\sqrt{3}},$$

$$a=\sqrt{\frac{T}{1+\sqrt{3}}}=\sqrt{\frac{T}{2}(\sqrt{3}-1)}.$$

Ex. 526. Find the base edge of a regular pyramid with a square base, having given the height h and the total surface T.

Let x = slant height.

$$x=\sqrt{\frac{a^2}{4}+h^2}.$$

$$T=2ax+a^2,$$

$$T=2a\sqrt{\frac{a^2}{4}+h^2}+a^2.$$

$$\frac{a^2}{4}+h^2=\frac{T^2-2a^2T+a^4}{4a^2},$$

$$a^4+4a^2h^2=T^2-2a^2T+a^4,$$

$$a^2(2T+4h^2)=T^2,$$

$$a^2=\frac{T^2}{2T+4h^2}=\frac{2T^2}{4(T+2h^2)},$$

$$a=\frac{T}{2}\sqrt{\frac{2}{T+2h^2}}.$$

Ex. 527. If the total surface of a right circular cylinder closed at both ends is a, and the radius of the base is r, what is the height of the cylinder?

$$a = 2\pi rh + 2\pi r^2.$$

$$2\pi rh = a - 2\pi r^2.$$

$$h = \frac{a}{2\pi r} - r.$$

Ex. 528. If the lateral surface of a right circular cylinder is a, and the volume is b, find the radius of the base and the height.

$$\begin{array}{r}a = 2\pi rh.\\ \underline{b = \ \ \pi r^2h.}\\ ar = 2\pi r^2h,\end{array}$$

and $$2b = 2\pi r^2h.$$

Hence $$ar = 2b,$$

and $$r = \frac{2b}{a}.$$

$$\therefore\ a = 2\pi\left(\frac{2b}{a}\right)h,$$

and $$h = \frac{a}{2\pi \times \frac{2b}{a}} = \frac{a^2}{4\pi b}.$$

Ex. 529. How many cubic yards of earth must be removed in constructing a tunnel 100 yards long, whose section is a semicircle with a radius of 10 feet?

$$V = \tfrac{1}{2}\pi r^2h = \frac{1}{\cancel{2}} \times \frac{22}{7} \times \frac{\overset{5}{\cancel{10}}}{3} \times \frac{10}{3} \times \frac{100}{1} = \frac{110000}{63} = 1746\tfrac{2}{63} \text{ cu. yds.}$$

Ex. 530. If the diameter of a well is 7 feet, and the water is 10 feet deep, how many gallons of water are there, reckoning 7½ gallons to the cubic foot?

$$V = \pi r^2h = \tfrac{22}{7} \times \tfrac{7}{2} \times \tfrac{7}{2} \times 10 \text{ cu. ft.}$$

$$= \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{2}} \times \frac{7}{\cancel{2}} \times \frac{\overset{5}{\cancel{10}}}{1} \times \frac{15}{2} \text{ gals.} = \frac{5775}{2} = 2887\tfrac{1}{2} \text{ gals.}$$

Ex. 531. When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 30 centimeters; find the volume of the body.

$$V = \pi r^2 h = \pi \times 30^2 \times 30 = \pi \times 27{,}000.$$

$$\begin{array}{r} 3.1416 \\ 27000 \\ \hline 219912000 \\ 62832 \\ \hline 84823.2000 \\ = 84823.2^{\text{ccm}}. \end{array}$$

Ex. 532. If the circumference of the base of a right circular cylinder is c, and the height h, find the volume V.

$$r = \frac{c}{2\pi}.$$

$$V = \pi r^2 h = \pi \frac{c^2}{4\pi^2} h = \frac{c^2 h}{4\pi}.$$

Ex. 533. Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V.

$$h = 2r.$$

$$V = \pi r^2 h = 2\pi r^3.$$

$$T = 2\pi r h + 2\pi r^2 = 4\pi r^2 + 2\pi r^2 = 6\pi r^2.$$

Hence
$$r^2 = \frac{T}{6\pi};\quad r = \sqrt{\frac{T}{6\pi}};\quad r^3 = \frac{T}{6\pi}\sqrt{\frac{T}{6\pi}}.$$

$$V = 2\pi \times \frac{T}{6\pi}\sqrt{\frac{T}{6\pi}} = \frac{T}{3}\sqrt{\frac{T}{6\pi}}.$$

Ex. 534. If the circumference of the base of a right circular cylinder is c, and the total surface is T, find the volume V.

$$V = \pi r^2 h.$$

$$T = 2\pi r^2 + 2\pi r h.$$

$$c = 2\pi r.$$

$$\therefore r = \frac{c}{2\pi}.$$

Hence
$$V = \frac{crh}{2} = \frac{c^2 h}{4\pi},$$

and $$T = cr + ch = \frac{c^2}{2\pi} + ch.$$

$$\therefore\ ch = T - \frac{c^2}{2\pi}.$$

$$\therefore \frac{c^2h}{4\pi} = \frac{cT}{4\pi} - \frac{c^3}{8\pi^2}.$$

$$\therefore\ V = \frac{cT}{4\pi} - \frac{c^3}{8\pi^2}.$$

$$\therefore\ V = \frac{c}{8\pi^2}(2\pi T - c^2).$$

Ex. 535. The slant height of a right circular cone is 2 feet. At what distance from the vertex must the slant height be cut by a plane parallel to the base, in order that the lateral surface may be divided into two equivalent parts?

If a denotes the slant height,

$$S = \pi ra = 2\pi r.$$

$$S : S' = 2 : 1.$$

$$\therefore\ a : a' = \sqrt{2} : 1.$$

$$2 : a' = \sqrt{2} : 1.$$

$$\therefore\ a' = \sqrt{2} = 1.414 \text{ ft.}$$

Ex. 536. The height of a right circular cone is equal to the diameter of its base; find the ratio of the area of the base to the lateral surface.

$$B = \pi r^2 = \tfrac{1}{4}\pi h^2.$$

The slant height $$a = \sqrt{h^2 + \frac{h^2}{4}} = \sqrt{\frac{5h^2}{4}} = \tfrac{1}{2}h\sqrt{5}.$$

$$S = \tfrac{1}{2}\pi h \times \tfrac{1}{2}h\sqrt{5} = \tfrac{1}{4}\pi h^2\sqrt{5}.$$

$$\therefore \frac{B}{S} = \frac{\frac{1}{4}\pi h^2}{\frac{1}{4}\pi h^2\sqrt{5}} = \frac{1}{\sqrt{5}} = \tfrac{1}{5}\sqrt{5}.$$

Ex. 537. What length of canvas ¾ of a yard wide is required to make a conical tent 12 feet in diameter and 8 feet high?

The slant height $a = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10.$

$$S = \pi ra = \frac{22}{7} \times \frac{\overset{2}{\cancel{6}}}{1} \times \frac{10}{1} \times \frac{1}{9} \times \frac{4}{\cancel{3}} = \frac{1760}{63}$$
$$= 27\tfrac{59}{63} \text{ yds.}$$

Ex. 538. The circumference of the base of a circular cone is $12\frac{1}{2}$ feet, and its height $8\frac{1}{4}$ feet; find its volume.

$$2\pi r = 12\tfrac{1}{2}.$$

Hence $$2 \times \tfrac{22}{7} r = \tfrac{25}{2}, \text{ and } r = \frac{\frac{25}{2}}{2 \times \frac{22}{7}} = \frac{\frac{25}{2}}{\frac{44}{7}}.$$

$$V = \tfrac{1}{3}\pi r^2 h = \frac{1}{\cancel{3}} \times \frac{\cancel{22}}{\cancel{7}} \times \frac{25}{2} \times \frac{25}{2} \times \frac{\cancel{7}}{\underset{2}{\cancel{44}}} \times \frac{7}{\underset{4}{\cancel{44}}} \times \frac{\overset{\cancel{3}}{\cancel{33}}}{4} = \frac{4375}{128}$$

$$= 34\tfrac{23}{128} \text{ cu. ft.} = 34 \text{ cu. ft. } 310\tfrac{1}{2} \text{ cu. in.}$$

Ex. 539. Given the total surface T of a right circular cone, and the radius r of the base; find the volume V.

If a denotes the slant height,

$$T = \pi r a + \pi r^2.$$

But $$a^2 = h^2 + r^2,$$

and $$a = \sqrt{h^2 + r^2}.$$

$$\therefore T = \pi r\sqrt{h^2 + r^2} + \pi r^2.$$

$$\sqrt{h^2 + r^2} = \frac{T - \pi r^2}{\pi r}.$$

$$h^2 + r^2 = \frac{T^2 - 2\pi r^2 T + \pi^2 r^4}{\pi^2 r^2}.$$

$$h^2 = \frac{T^2 - 2\pi r^2 T}{\pi^2 r^2},$$

$$h = \frac{1}{\pi r}\sqrt{T(T - 2\pi r^2)}.$$

But $$V = \tfrac{1}{3}\pi r^2 h$$

$$= \tfrac{1}{3}\pi r^2 \times \frac{1}{\pi r}\sqrt{T(T - 2\pi r^2)}$$

$$= \frac{r}{3}\sqrt{T(T - 2\pi r^2)}.$$

Ex. 540. Given the total surface T of a right circular cone, and the lateral surface, S; find the volume V.

If a denotes the slant height,

$$T = \pi r a + \pi r^2 = \pi r(a + r),$$

and $$S = \pi r a.$$

But $$a^2 = h^2 + r^2.$$

$$\therefore h = \sqrt{a^2 - r^2}.$$

$$\begin{aligned}V &= \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi r^2\sqrt{a^2 - r^2}\\ &= \tfrac{1}{3}\pi r^2\sqrt{(a+r)(a-r)}\\ &= \tfrac{1}{3}\sqrt{r(a+r)(\pi r^2)(\pi r)(a-r)}\\ &= \frac{1}{3}\sqrt{\frac{[\pi r(a+r)][\pi r^2][\pi r(a-r)]}{\pi}}\\ &= \frac{1}{3}\sqrt{\frac{T(T-S)(2S-T)}{\pi}}\end{aligned}$$

Ex. 541. How many square feet of tin will be required to make a funnel if the diameters of the top and bottom are to be 28 inches and 14 inches respectively, and the height 24 inches?

$$\begin{aligned}a &= \sqrt{24^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25.\\ S &= \pi a(r + r') = \frac{22}{\cancel{7}} \times 25 \times \overset{3}{\cancel{21}}\\ &= 1650 \text{ sq. in.}\\ &= 11 \text{ sq. ft. } 66 \text{ sq. in.}\end{aligned}$$

Ex. 542. Find the expense of polishing the curved surface of a marble column in the shape of the frustum of a right cone whose slant height is 12 feet, and the radii of the circular ends are 3 feet 6 inches and 2 feet 4 inches respectively, at 60 cents a square foot.

$$S = \pi a(r + r') = \tfrac{22}{7} \times \tfrac{35}{6} \times \tfrac{12}{1}.$$

$$\text{Cost} = \frac{22}{\cancel{7}} \times \frac{\overset{5}{\cancel{35}}}{\cancel{6}} \times \frac{\overset{2}{\cancel{12}}}{1} \times \frac{\overset{3}{\cancel{\$60}}}{\underset{5}{\cancel{100}}} = \$132.$$

Ex. 543. The slant height of the frustum of a regular square pyramid is 20 feet, the length of each side of its base 40 feet, of each side of its top 16 feet; find its volume.

$$\begin{aligned}h &= \sqrt{20^2 - 12^2} = \sqrt{400 - 144} = \sqrt{256} = 16.\\ V &= \tfrac{1}{3}h(B + B' + \sqrt{BB'}) = \tfrac{1}{3}h(40^2 + 16^2 + 40 \times 16)\\ &= \tfrac{1}{3}h(1600 + 256 + 640)\\ &= \frac{1}{\cancel{3}} \times 16 \times \overset{832}{\cancel{2496}}\\ &= 13{,}312 \text{ cu. ft.}\end{aligned}$$

Ex. 544. If the bases of the frustum of a pyramid are two regular hexagons whose sides are 1 foot and 2 feet respectively, and the volume of the frustum is 12 cubic feet, find its height.

$$\sqrt{1^2 - (\tfrac{1}{2})^2} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3} = 0.866.$$

$$B' = 6 \times \tfrac{1}{2}(1 \times 0.866) = 2.598.$$

$$B : B' = 1^2 : 2^2 = 1 : 4.$$

$$\therefore B = 4 \times 2.598 = 10.392.$$

$$V = \tfrac{1}{3} h(B + B' + \sqrt{BB'}).$$

$$12 = \tfrac{1}{3} h(2.598 + 10.392 + 5.196).$$

$$36 = 18.186\,h.$$

$$h = \frac{36}{18.186} = 1.98 \text{ ft.}$$

Ex. 545. The frustum of a right circular cone is 14 feet high, and has a volume of 924 cubic feet. Find the radii of its bases if their sum is 9 feet.

Let r and $(9 - r)$ be the radii.

$$V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr').$$

$$924 = \tfrac{1}{3} \times \tfrac{22}{7} \times 14\,[r^2 + (9 - r)^2 + r(9 - r)]$$

$$= \tfrac{44}{3}(r^2 + 81 - 18r + r^2 + 9r - r^2).$$

$$63 = r^2 - 9r + 81.$$

$$r^2 - 9r = -18.$$

$$4r^2 - (\ ) + 81 = 9,$$

$$2r - 9 = \pm 3,$$

$$2r = 12 \text{ or } 6,$$

$$r = 6 \text{ or } 3,$$

$$9 - r = 3 \text{ or } 6.$$

Therefore the radii are 6 ft. and 3 ft.

Ex. 546. From a right circular cone whose slant height is 30 feet, and circumference of whose base is 10 feet, there is cut off by a plane parallel to the base a cone whose slant height is 6 feet. Find the convex surface and the volume of the frustum.

If the circumference of the section is denoted by c',

$$c' = \tfrac{6}{30} \text{ of } 10 = 2.$$

Then $$S = \tfrac{1}{2} \times 24(10 + 2) = 12 \times 12 = 144 \text{ sq. ft.}$$

Now $$2\pi r = 10, \text{ and } 2\pi r' = 2.$$

$$\therefore r = \frac{5}{\pi}, \quad \text{and} \quad r' = \frac{1}{\pi}.$$

$$\therefore h = \sqrt{24^2 - \left(\frac{4}{\pi}\right)^2} = \sqrt{576 - \frac{16}{\pi^2}} = 4\sqrt{36 - \left(\frac{7}{22}\right)^2} = \frac{4\sqrt{17375}}{22}$$

$$= \tfrac{2}{11} \times 131.814 = 23.966.$$

But $$V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr').$$

$$= \frac{1}{3} \times \frac{22}{7} \times 23.966 \left(\frac{25}{\pi^2} + \frac{1}{\pi^2} + \frac{5}{\pi^2}\right)$$

$$= \frac{1}{3} \times \frac{\cancel{22}}{\cancel{7}} \times 31 \times \frac{\cancel{7}}{\cancel{22}} \times \frac{7}{22} \times 23.966$$

$$= \tfrac{217}{66} \times 23.966 = 78.8 \text{ cu. ft.}$$

Ex. 547. Find the difference between the volume of the frustum of a pyramid whose bases are squares, measuring 8 feet and 6 feet respectively on a side, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

$$V = \tfrac{1}{3}h(B + B' + \sqrt{BB'}) = \tfrac{1}{3}h(64 + 36 + 48) = \tfrac{1}{3}h \times 148 = 49\tfrac{1}{3}h.$$

$$V' = 49h.$$

$$V - V' = \tfrac{1}{3}h \text{ cu. ft.}$$

Ex. 548. A Dutch windmill in the shape of the frustum of a right cone is 12 meters high. The outer diameters at the bottom and the top are 16 meters and 12 meters, the inner diameters 12 meters and 10 meters, respectively. How many cubic meters of stone were required to build it?

$$V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr') = \tfrac{1}{3}\pi\, 12(64 + 36 + 48) = 4\pi \times 148 = 592\pi.$$

$$V' = \tfrac{1}{3}\pi\, 12(36 + 25 + 30) = 4\pi \times 91 = 364\pi.$$

$$592\pi - 364\pi = 228\pi = 716.2848^{\text{cbm}}.$$

$$\begin{array}{r} 3.1416 \\ \underline{228} \\ 251328 \\ 62832 \\ \underline{62832} \\ 716.2848 \end{array}$$

Ex. 549. The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is 180 feet, and its upper and lower bases are squares whose sides are 10 feet and 16 feet respectively. The flue is throughout a square whose side is 7 feet. How many cubic feet of material does the chimney contain?

$$\begin{aligned} V = \tfrac{1}{3} h (B + B' + \sqrt{BB'}) &= \tfrac{1}{3} \times 180 (256 + 100 + 160) \\ &= 60 \times 516 = 30960 \text{ cu. ft.} \end{aligned}$$

$$180 \times 7^2 = 180 \times 49 = 8820 \text{ cu. ft.}$$

$$30960 - 8820 = 22,140 \text{ cu. ft.}$$

Ex. 550. Find the volume V of the frustum of a cone of revolution, having given the slant height a, the height h, and the convex surface S.

$$S = \pi a r + \pi a r'.$$

$$r' = r - \sqrt{a^2 - h^2}.$$

$$S = \pi a r + \pi a r - \pi a \sqrt{a^2 - h^2}.$$

$$S = 2\pi a r - \pi a \sqrt{a^2 - h^2}.$$

$$r = \frac{S}{2\pi a} + \tfrac{1}{2}\sqrt{a^2 - h^2} \text{ and } r' = \frac{S}{2\pi a} - \tfrac{1}{2}\sqrt{a^2 - h^2}.$$

$$\begin{aligned} V = \tfrac{1}{3}\pi h\Big[\frac{S^2}{4\pi^2 a^2} &+ \frac{S\sqrt{a^2 - h^2}}{2\pi a} + \tfrac{1}{4}(a^2 - h^2) \\ &+ \frac{S^2}{4\pi^2 a^2} - \frac{S\sqrt{a^2 - h^2}}{2\pi a} + \tfrac{1}{4}(a^2 - h^2) \\ &+ \frac{S^2}{4\pi^2 a^2} \qquad\qquad - \tfrac{1}{4}(a^2 - h^2)\Big]. \end{aligned}$$

$$\therefore V = \tfrac{1}{3}\pi h\left[\frac{3S^2}{4\pi^2 a^2} + \tfrac{1}{4}(a^2 - h^2)\right].$$

Ex. 551. A cube whose edge is 12 inches long is transformed into a right prism whose base is a rectangle 16 inches long and 12 inches wide. Find the height of the prism, and the difference between its total surface and the surface of the cube.

$$V = 12^3 = 1728 \text{ cu. in.}$$

$$B = 16 \times 12 = 192 \text{ sq. in.}$$

$$1728 \div 192 = 9 \text{ in. (height).}$$

$$T = 6 \times 12 \times 12 = 864 \text{ sq. in.}$$

$$\begin{aligned} T' &= 2(16 \times 12 + 16 \times 9 + 12 \times 9) \\ &= 2(192 + 144 + 108) \\ &= 2 \times 444 = 888 \text{ sq. in.} \end{aligned}$$

$$T' - T = 888 - 864 = 24 \text{ sq. in.}$$

Ex. 552. The dimensions of a rectangular parallelopiped are a, b, c. Find (i.) the height of an equivalent right circular cylinder having a for the radius of its base; (ii.) the height of an equivalent right circular cone having a for the radius of its base.

$$\text{(i.)}\quad V = abc = \pi a^2 h.$$

$$\therefore h = \frac{abc}{\pi a^2} = \frac{bc}{\pi a}.$$

$$\text{(ii.)}\quad V' = \tfrac{1}{3}\pi r^2 h.$$

$$abc = \tfrac{1}{3}\pi a^2 h.$$

$$h = \frac{3\,abc}{\pi a^2} = \frac{3\,bc}{\pi a}.$$

Ex. 553. A regular pyramid 12 feet high is transformed into a regular prism with an equivalent base; what is the height of the prism?

$$V = \tfrac{1}{3}Bh = \tfrac{1}{3}B \times 12.$$

$$V' = Bh.$$

$$\tfrac{1}{3}B \times 12 = Bh.$$

$$h = \frac{4\,B}{B} = 4 \text{ ft.}$$

Ex. 554. The diameter of a cylinder is 14 feet, and its height is 8 feet; find the height of an equivalent right prism the base of which is a square with a side 4 feet long.

$$V = \pi r^2 h$$

$$= \frac{22}{7} \times \frac{7^2}{1} \times \frac{8}{1} = 1232 \text{ cu. ft.}$$

$$V' = Bh.$$

$$1232 = 4^2 h = 16\,h.$$

$$h = \tfrac{1232}{16} = 77 \text{ ft.}$$

Ex. 555. If one edge of a cube is a, what is the height h of an equivalent right circular cylinder whose diameter is b?

$$V = a^3.$$

$$V' = \pi r^2 h.$$

$$r = \frac{b}{2}.$$

$$a^3 = \pi \left(\frac{b}{2}\right)^2 h.$$

$$4\,a^3 = \pi b^2 h.$$

$$h = \frac{4\,a^3}{\pi b^2}.$$

Ex. 556. The heights of two equivalent right circular cylinders are as 4 : 9. The diameter of the first is 6 feet; what is the diameter of the other?

$$V = \pi r^2 h = \pi 3^2 h = 9\pi h.$$

$$V' = \pi r'^2 \left(\frac{9h}{4}\right) = \frac{9\pi r'^2 h}{4}.$$

$$\frac{9\pi r'^2 h}{4} = 9\pi h.$$

$$r'^2 = 4, \qquad r' = 2.$$

$$\text{Diameter} = 2r' = 4 \text{ ft.}$$

Ex. 557. A right circular cylinder 6 feet in diameter is equivalent to a right circular cone 7 feet in diameter. If the height of the cone is 8 feet, what is the height of the cylinder?

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi(\tfrac{7}{2})^2 \times 8$$

$$= \tfrac{1}{3}\pi \times \frac{49}{\cancel{4}} \times \overset{2}{\cancel{8}} = \frac{98\pi}{3}.$$

$$V' = \pi r^2 h = \pi 3^2 h = 9\pi h.$$

$$9\pi h = \frac{98\pi}{3}.$$

$$h = \tfrac{98}{27} = 3.63 \text{ ft.}$$

Ex. 558. The frustum of a regular four-sided pyramid is 6 feet high, and the sides of its bases are 5 feet and 8 feet respectively. What is the height of an equivalent regular pyramid whose base is a square with a side 12 feet long?

$$V = \tfrac{1}{3}h(B + B' + \sqrt{BB'})$$

$$= \tfrac{1}{3} \times 6(25 + 64 + 40)$$

$$= 2 \times 129 = 258 \text{ cu. ft.}$$

$$V' = \tfrac{1}{3}Bh.$$

$$258 = \tfrac{1}{3} \times 144h.$$

$$48h = 258.$$

$$h = 5\tfrac{3}{8} \text{ ft.}$$

Ex. 559. The frustum of a cone of revolution is 5 feet high, and the diameters of its bases are 2 feet and 3 feet respectively; find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to its bases, and equidistant from the bases.

$$V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr')$$

$$= \tfrac{1}{3}\pi \times 5(1 + \tfrac{9}{4} + \tfrac{3}{2})$$

$$= \tfrac{5}{3} \times \tfrac{19}{4} \times \pi$$

$$= \frac{95\pi}{12} = 7.9167\pi.$$

$$V' = \pi r^2 h.$$

$$7.9167\,\pi = \pi \tfrac{25}{16}\,h.$$

$$h = \frac{16 \times 7.9167}{25} = 5.0667 \text{ ft.}$$

Ex. 560. Find the edge of a cube equivalent to a regular tetrahedron whose edge measures 3 inches.

$$\text{Altitude of } \Delta = \sqrt{9 - \tfrac{9}{4}} = \sqrt{\tfrac{27}{4}} = \tfrac{3}{2}\sqrt{3}.$$

$$\tfrac{2}{3} \times \tfrac{3}{2}\sqrt{3} = \sqrt{3}.$$

$$h = \sqrt{3^2 - (\sqrt{3})^2} = \sqrt{9 - 3} = \sqrt{6}.$$

$$B = \tfrac{1}{2} \times 3 \times \tfrac{3}{2}\sqrt{3} = \tfrac{9}{4}\sqrt{3}.$$

$$V = \tfrac{1}{3} \times \tfrac{9}{4}\sqrt{3} \times \sqrt{6} = \frac{1}{\cancel{3}} \times \frac{9}{4} \times \cancel{3} \times \sqrt{2}$$

$$= \tfrac{9}{4}\sqrt{2} = \tfrac{9}{4} \times 1.41421 = 3.181973 \text{ cu. in.}$$

$$\text{An edge} = \sqrt[3]{3.181973} = 1.47 \text{ in.}$$

Ex. 561. Find the edge of a cube equivalent to a regular octahedron whose edge measures 3 inches.

Let h denote the altitude of each of the two pyramids into which the octahedron can be divided, and V the volume of each pyramid.

Then $$h = \sqrt{9 - \tfrac{9}{2}} = \sqrt{\tfrac{9}{2}} = \tfrac{3}{2}\sqrt{2},$$

and $$V = \tfrac{1}{3} \times 3^2 \times \tfrac{3}{2}\sqrt{2}.$$

$$\therefore 2V = \cancel{2} \times \frac{1}{\cancel{3}} \times \frac{9}{1} \times \frac{\cancel{3}}{\cancel{2}}\sqrt{2} = 9\sqrt{2}$$

$$= 9 \times 1.14121 = 12.72789 \text{ cu. in.}$$

$$\text{An edge} = \sqrt[3]{12.72789} = 2.33 \text{ in.}$$

Ex. 562. The dimensions of a trunk are 4 feet, 3 feet, 2 feet. What are the dimensions of a trunk similar in shape that will hold four times as much?

$1 : 4 = 4^3 : x^3.$	$1 : 4 = 3^3 : x^3.$	$1 : 4 = 2^3 : x^3.$
$x^3 = 256,$	$x^3 = 108,$	$x^3 = 32,$
$x = 6.35$ ft.	$x = 4.76$ ft.	$x = 3.17$ ft.

Ex. 563. By what number must the dimensions of a cylinder be multiplied in order to obtain a similar cylinder (i.) whose surface shall be n times that of the first; (ii.) whose volume shall be n times that of the first?

$$\text{(i.) } \sqrt{n} \text{ times.} \qquad\qquad \text{(ii.) } \sqrt[3]{n} \text{ times.}$$

Ex. 564. A pyramid is cut by a plane which passes midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

$$2^3 : 1 = 8 : 1.$$

Ex. 565. The height of a regular hexagonal pyramid is 36 feet, and one side of the base is 6 feet. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?

$$20 : 1 = 36^3 : h^3. \qquad\qquad 20 : 1 = 6^3 : a^3.$$

$$h^3 = \frac{36^3}{20} = 2332.8, \qquad\qquad a^3 = 10.8,$$

$$h = 13.2625 \text{ ft.} \qquad\qquad a = 2.2104 \text{ ft.}$$

Ex. 566. The length of one of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

$$V : V' = 4^3 : x^3.$$

$$2 : 1 = 64 : x^3.$$

$$x^3 = 32,$$

$$x = 3.17^{\text{m}}.$$

Ex. 567. The length of a lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base, which divide the pyramid into three equivalent parts?

$$V : V' = a^3 : a'^3. \qquad\qquad V : V' = a^3 : a'^3.$$

$$3 : 1 = a^3 : a'^3. \qquad\qquad 3 : 2 = a^3 : a'^3.$$

$$3a'^3 = a^3, \qquad\qquad 3a'^3 = 2a^3,$$

$$a' = a\sqrt[3]{\tfrac{1}{3}}. \qquad\qquad a' = a\sqrt[3]{\tfrac{2}{3}}.$$

Ex. 568. The length of a lateral edge of a pyramid is a. At what distance from the vertex will this edge be cut by a plane parallel to the base, and dividing the pyramid into two parts which are to each other as 3 : 4?

$$\begin{aligned} V : V' &= a^3 : a'^3. \\ 7 : 3 &= a^3 : a'^3. \\ 7a'^3 &= 3a^3, \\ a' &= a\sqrt[3]{\tfrac{3}{7}}. \end{aligned}$$

Ex. 569. The volumes of two similar cones are 54 cubic feet and 432 cubic feet. The height of the first is 6 feet; what is the height of the other?

$$\begin{aligned} \sqrt[3]{54} : \sqrt[3]{432} &= 6 : x. \\ 1 : \sqrt[3]{8} &= 6 : x. \\ 1 : 2 &= 6 : x. \\ x &= 12 \text{ ft.} \end{aligned}$$

Ex. 570. In each of two right circular cylinders the diameter is equal to the height. The volume of one is $\frac{3}{4}$ that of the other. What is the ratio of their heights?

$$\begin{aligned} V &= \pi r^2 h = 2\pi r^3. \\ V : V' &= 2\pi r^3 : 2\pi r'^3. \\ 1 : \tfrac{3}{4} &= r^3 : r'^3. \\ r : r' &= 1 : \sqrt[3]{\tfrac{3}{4}} = 1 : 0.90856. \end{aligned}$$

Ex. 571. Find the dimensions of a right circular cylinder $\frac{15}{16}$ as large as a similar cylinder whose height is 20 feet, and diameter 10 feet.

$$\begin{array}{ll} h : h' = \sqrt[3]{V} : \sqrt[3]{V'} & 2r : 2r' = \sqrt[3]{V} : \sqrt[3]{V'}. \\ 20 : h' = 1 : \sqrt[3]{\tfrac{15}{16}}. & 10 : 2r' = 1 : \sqrt[3]{\tfrac{15}{16}}. \\ 20 : h' = 1 : 0.97871, & 10 : 2r' = 1 : 0.97871, \\ \quad h' = 20 \times 0.97871 & \quad 2r' = 10 \times 0.97871 \\ \quad\quad = 19.5742 \text{ ft.} & \quad\quad = 9.7871 \text{ ft.} \end{array}$$

Ex. 572. The height of a cone of revolution is h, and the radius of its base is r. What are the dimensions of a similar cone three times as large?

$$\begin{array}{ll} h : h' = \sqrt[3]{V} : \sqrt[3]{V'}. & r : r' = \sqrt[3]{V} : \sqrt[3]{V'}. \\ h : h' = 1 : \sqrt[3]{3}, & r : r' = 1 : \sqrt[3]{3}, \\ \quad h' = h\sqrt[3]{3}. & \quad r' = r\sqrt[3]{3}. \end{array}$$

Ex. 573. The height of the frustum of a right cone is $\frac{2}{5}$ the height of the entire cone. Compare the volumes of the frustum and the entire cone.

Let V be the volume of the cone,

V' be the volume cut off,

V'' be the volume of the frustum.

$$V : V' = h^3 : h'^3$$

$$= 5^3 : 3^3 = 125 : 27.$$

$$V'' : V = 125 - 27 : 125$$

$$= 98 : 125.$$

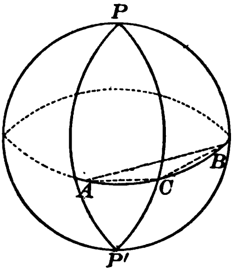
Ex. 574. The frustum of a pyramid is 8 feet high, and two homologous edges of its bases are 4 feet and 3 feet respectively. Compare the volume of the frustum and that of the entire pyramid.

$$V : V' = 4^3 : 3^3 = 64 : 27.$$

$$V'' : V = 64 - 27 : 64 = 37 : 64.$$

Book VIII.

Ex. 575. Every point in a great circle which bisects a given arc of a great circle at right angles, is equidistant from the extremities of the given arc.



Let AB be any given arc, and PCP' be the great ⊙ whose circumference bisects the arc AB at right angles.

To prove that every point in the great ⊙ PCP' is equidistant from A and B.

Proof. Chord $AC =$ chord BC. (§ 230)

⊙ PCP' passes through the poles of arc AB. (§ 696)

But the poles of arc AB are equidistant from A and B, the distances being measured on straight lines.

Hence the plane of the ⊙ PCP' is ⊥ to the chord AB at its middle point. (§§ 459, 482)

Therefore every point in the great ⊙ PCP' is equidistant from the points A and B.

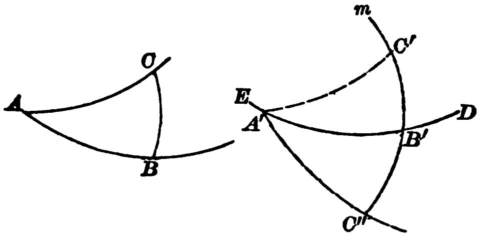
Ex. 576. The radius of a sphere is 4 inches. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 inches. Find the area of this circle.

Let O be the centre of the sphere, P the pole, and A a point in the ⊙.

$$\begin{aligned}\text{Area of } \triangle POA &= \sqrt{5\tfrac{1}{2} \times 1\tfrac{1}{2} \times 1\tfrac{1}{2} \times 2\tfrac{1}{2}}\\ &= \tfrac{1}{4}\sqrt{11 \times 3 \times 3 \times 5}\\ &= \tfrac{1}{4}\sqrt{495}\\ &= \tfrac{1}{4} \times 22.25 = 5.56.\end{aligned}$$

Also,
$$\text{area of } \triangle = \tfrac{1}{2} \times 4 \times h = 2h.$$
$$\therefore 5.56 = 2h.$$
$$r = h = 2.78.$$
$$\begin{aligned}\text{Area } \odot &= \pi r^2\\ &= \tfrac{22}{7} \times 2.78^2 = 24.289 \text{ sq. in.}\end{aligned}$$

Ex. 577. At a given point in a given arc of a great circle, to construct a spherical angle equal to a given spherical angle.



Let A' be any point in the arc ED, and BAC any given $\angle$. It is required to construct an $\angle$ equal to BAC having A' as its vertex, and $A'D$ as one side.

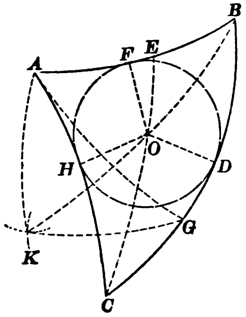
CONSTRUCTION. With A as a pole and a quadrant as a radius arc, draw the arc BC. With A' as a pole, draw the arc of a great $\odot$ $B'm$. On this arc lay off $B'C'$ equal to BC, and through A' and C' draw the arc of a great $\odot$ $A'C'$; then will $B'A'C'$ be the $\angle$ required.

PROOF. The arcs BC and $B'C'$ are the measures of the $\angle$s A and A', respectively (§ 722). But $BC = B'C'$, by construction; hence $A = A'$.

Another $\angle$ $B'A'C''$ might be constructed below the line $A'D$.

Ex. 578. To inscribe a circle in a given spherical triangle.

CONSTRUCTION. Let ABC be any spherical $\triangle$. From B as a pole, and with BA as radius arc, describe the arc AG. From A and G as poles, and with a radius arc greater than $\frac{1}{2}AG$, describe arcs intersecting in K. Through K and B draw an arc of a great $\odot$.



In like manner draw CE. From O, the intersection of BK and CE, let fall the arc of a great $\odot$ OD $\perp$ to the side BC.

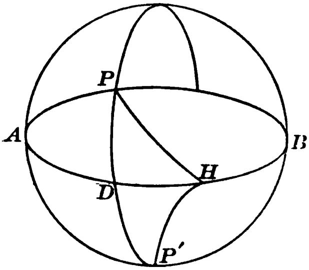
From O as a pole, and with OD as radius arc, draw the $\odot$ DFH. DFH is the $\odot$ required.

PROOF. Draw the arcs of great $\odot$s KA and KG. Lay off $BF = BD$, $CH = CD$, and draw the arcs of great $\odot$s OF and OH. Since, by construction, the $\triangle$s BAK and BGK are mutually equilateral, they are therefore mutually equiangular, and $\angle OBF = \angle OBD$. $\therefore$ $\triangle$s OBF and OBD are equivalent. (§ 749)

Hence OF is $\perp$ to AB and equal to OD.

In like manner, we may prove that OH is $\perp$ to AC and equal to AD.

It remains to prove that the $\perp$s OD, OF, and OH measure the shortest distances from O to the sides CB, AB, and AC, respectively. To do this, let PD be the less of the two $\perp$s from P to the arc ADB, and let PH be any other arc of a great ⊙ from P to ADB.



Then $\quad PD < PH$.

For $\quad HP = HP'$. $\quad$ (Ex. 575)

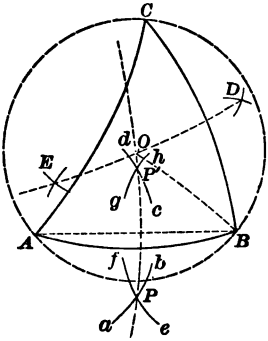
$PD + P'D < PH + HP'$. $\quad$ (§ 699)

$\therefore 2\,PD < 2\,PH$, or $PD < PH$.

Hence the ⊙ described lies within the given △ ABC, and touches its sides at D, F, and H, respectively.

Ex. 579. To circumscribe a circle about a given spherical triangle.

Construction. Let ABC be any spherical △. From A as pole, and with a radius arc greater than ½ AB, draw the arcs ab and cd. From B as pole, and with the same radius arc, draw the arcs ef and gh, intersecting the first arcs in P and P'. Through P and P' draw the arc of a great ⊙ $OP'P$.



In like manner draw the arc of a great ⊙ DE. From O, the intersection of PP' and ED, as pole, and OB as radius arc, draw the ⊙ BCA. BCA is the ⊙ required.

Proof. Since by construction P and P' are equidistant from A and B, the distances being measured on arcs of great circles, they are equidistant from A and B, distances being measured on straight lines.

The centre of the sphere is also equidistant from A and B. Hence the plane of the arc PP' is perpendicular to the chord AB at its middle point (§§ 459, 482).

Therefore O is equidistant from A and B, distances being measured on straight lines, and therefore equidistant, distances being measured on arcs of great ⓢ (§ 482). In like manner O is equidistant from B and C.

Hence the circumference of the ⊙ described from O as a pole, and with OB as radius arc, will pass through A, B, and C.

Ex. 580. Given a spherical triangle whose sides are 60°, 80°, and 100°; find the angles of its polar triangle.

The $\measuredangle$ of polar $\triangle$ are 120°, 100°, and 80°. (§ 735)

Ex. 581. Given a spherical triangle whose angles are 70°, 75°, and 95°; find the sides of its polar triangle.

The sides of polar $\triangle$ are 110°, 105°, and 85°. (§ 735)

Ex. 582. Given two mutually equiangular triangles on spheres whose radii are 12 inches and 20 inches respectively; find the ratio of two homologous sides of these triangles. (See note, page 362.)

$$12 : 20 = 3 : 5.$$

Ex. 583. Find the area of the surface of a sphere whose radius is 6 inches.

$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times 6^2$$
$$= \frac{4 \times 22 \times 36}{7} = \frac{3168}{7} = 452\tfrac{4}{7} \text{ sq. in.}$$

Ex. 584. Find the area of a zone if its altitude is 3 inches, and the radius of the sphere is 6 inches.

$$Z = 3 \times 2\pi R = 3 \times 2 \times \tfrac{22}{7} \times 6 = \tfrac{792}{7} = 113\tfrac{1}{7} \text{ sq. in.} \qquad (§\ 766)$$

Ex. 585. Given the radius of a sphere 10 inches; find the area of a lune whose angle is 30°.

$$L : S = 30° : 360° = 1 : 12.$$
$$S = 4\pi R^2 = 4 \times \frac{22}{7} \times \frac{10^2}{1} = \frac{8800}{7}.$$
$$L : \tfrac{8800}{7} = 1 : 12.$$
$$L = \tfrac{8800}{84} = 104\tfrac{16}{21} \text{ sq. in.}$$

Ex. 586. Given the diameter of a sphere 16 inches; find the area of a lune whose angle is 75°.

$$L : S = 75° : 360° = 5 : 24.$$
$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times 64 = \tfrac{5632}{7} = 804\tfrac{4}{7}.$$
$$L : \tfrac{5632}{7} = 5 : 24.$$
$$24\,L = \tfrac{28160}{7}.$$
$$L = \frac{\overset{3520}{\cancel{28160}}}{7 \times \underset{3}{\cancel{24}}} = \frac{3520}{21} = 167\tfrac{13}{21} \text{ sq. in.}$$

Ex. 587. What part of the surface of a sphere is a triangle whose angles are 120°, 100°, and 95°? What is its area in square inches, if the radius of the sphere is 6 inches?

$$E = 120° + 100° + 95° - 180° = 135°.$$

$$\Delta : S = E : 720. \qquad (§\ 776)$$

$$\Delta : S = 135 : 720 = 3 : 16.$$

$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times \tfrac{36}{1} = \tfrac{3168}{7}.$$

$$\Delta : \tfrac{3168}{7} = 3 : 16.$$

$$\Delta = \frac{\overset{594}{\cancel{9504}}}{7 \times \cancel{16}} = \frac{594}{7} = 84\tfrac{6}{7} \text{ sq. in.}$$

Ex. 588. Find the area of a spherical triangle whose angles are 100°, 120°, 140°, if the diameter of the sphere is 16 inches.

$$E = 100° + 120° + 140° - 180° = 180°.$$

$$\Delta : S = E : 720.$$

$$S = \tfrac{5632}{7}. \qquad (\text{Ex. } 586)$$

$$\Delta : \tfrac{5632}{7} = 180 : 720 = 1 : 4.$$

$$\Delta = \frac{5632}{7 \times 4} = \frac{1408}{7} = 201\tfrac{1}{7} \text{ sq. in.}$$

Ex. 589. If the radii of two spheres are 6 inches and 4 inches respectively, and the distance between their centres is 5 inches, what is the area of the circle of intersection of these spheres?

$$\text{Area} = \sqrt{\tfrac{15}{2} \times \tfrac{3}{2} \times \tfrac{5}{2} \times \tfrac{7}{2}} = \tfrac{15}{4}\sqrt{7}.$$

$$\text{Area} = \tfrac{5}{2} R.$$

$$\tfrac{5}{2} R = \tfrac{15}{4}\sqrt{7}.$$

$$R = \frac{\cancel{2}}{\cancel{5}} \times \frac{\overset{3}{\cancel{15}}}{\underset{2}{\cancel{4}}} \sqrt{7} = \tfrac{3}{2}\sqrt{7}.$$

$$\text{Area of circle} = \pi R^2 = \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{9}{\underset{2}{\cancel{4}}} \times \cancel{7} = \frac{99}{2} = 49\tfrac{1}{2} \text{ sq. ft.}$$

Ex. 590. Find the radius of the circle determined in a sphere of 5 inches diameter by a plane 1 inch from the centre.

$$R = \sqrt{(2\tfrac{1}{2})^2 - 1^2} = \tfrac{1}{2}\sqrt{25 - 4} = \tfrac{1}{2}\sqrt{21} \text{ in.}$$

Ex. 591. If the radii of two concentric spheres are R and R', and if a plane is drawn tangent to the interior sphere, what is the area of the section made in the other sphere?

$$r = \sqrt{R^2 - R'^2}.$$

$$\text{Area} = \pi r^2 = \pi(R^2 - R'^2).$$

Ex. 592. Two points A and B are 8 inches apart. Find the locus in space of a point 5 inches from A and 7 inches from B.

Let R denote the ⊥ from the point C to AB.

Then $$R = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)} \qquad \text{(Ex. 222)}$$

$$= \tfrac{2}{8}\sqrt{10 \times 3 \times 5 \times 2} = \tfrac{1}{4} \text{ of } 10\sqrt{3} = \tfrac{5}{2}\sqrt{3}.$$

Hence the locus is the circumference described with R as radius and the foot of R as centre.

Ex. 593. The radii of two parallel sections of the same sphere are a and b respectively, and the distance between these sections is d; find the radius of the sphere.

$$OA = \sqrt{r^2 - a^2}; \quad OB = \sqrt{r^2 - b^2}.$$

$$OA - OB = d = \sqrt{r^2 - a^2} - \sqrt{r^2 - b^2}.$$

$$\therefore\ d + \sqrt{r^2 - b^2} = \sqrt{r^2 - a^2}.$$

$$d^2 + 2d\sqrt{r^2 - b^2} + r^2 - b^2 = r^2 - a^2.$$

$$2d\sqrt{r^2 - b^2} = b^2 - a^2 - d^2.$$

$$4d^2r^2 - 4b^2d^2 = a^4 + b^4 + d^4 - 2a^2b^2 - 2b^2d^2 + 2a^2d^2.$$

$$4d^2r^2 = a^4 + b^4 + d^4 + 2b^2d^2 + 2a^2d^2 - 2a^2b^2.$$

$$r = \frac{1}{2d}\sqrt{a^4 + b^4 + d^4 + 2b^2d^2 + 2a^2d^2 - 2a^2b^2}.$$

Ex. 594. Find the area of a spherical quadrangle whose angles are 170°, 139°, 126°, and 141°, if the radius of the sphere is 10 inches.

$$E = T - (n-2)\,180°.$$

$$T = 170° + 139° + 126° + 141° = 576°.$$

$$E = 576° - 2 \times 180° = 576° - 360° = 216°.$$

$$\frac{\text{Area}}{S} = \frac{216}{720} = \frac{3}{10}.$$

$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times 10^2 = \tfrac{8800}{7}.$$

$$\frac{\text{Area}}{\frac{8800}{7}} = \frac{3}{10}.$$

$$\text{Area} = \tfrac{3}{10} \times \tfrac{8800}{7} = \tfrac{2640}{7} = 377\tfrac{1}{7} \text{ sq. in.}$$

$$= 2 \text{ sq. ft. } 89 \text{ sq. in.}$$

Ex. 595. Find the area of a spherical pentagon whose angles are 122°, 128°, 131°, 160°, 161°, if the surface of the sphere is 150 square feet.

$$E = T - (n-2)\,180°.$$

$$T = 122° + 128° + 131° + 160° + 161° = 702°.$$

$$E = 702° - 3 \times 180° = 162°.$$

$$\frac{\text{Area}}{S} = \frac{162}{720} = \frac{9}{40}.$$

$$\frac{\text{Area}}{150} = \frac{9}{40}.$$

$$\text{Area} = \tfrac{1350}{40} = 33\tfrac{3}{4}\text{ sq. ft.} = 33\text{ sq. ft. }108\text{ sq. in.}$$

Ex. 596. Find the area of a spherical hexagon whose angles are 96°, 110°, 128°, 136°, 140°, 150°, if the circumference of a great circle of the sphere is 10 inches.

$$E = T - (n-2)\,180°.$$

$$T = 96° + 110° + 128° + 136° + 140° + 150° = 760°.$$

$$E = 760° - 4 \times 180° = 40°.$$

$$S = 4\pi R^2 = 4\pi\left(\frac{10}{2\pi}\right)^2 = \overset{50}{\cancel{100}} \times \frac{7}{\underset{11}{\cancel{22}}} = \frac{350}{11}\text{ sq. in.}$$

$$\frac{\text{Area}}{S} = \frac{E}{720}.$$

$$\frac{\text{Area}}{\tfrac{350}{11}} = \frac{40}{720} = \frac{1}{18}.$$

$$\text{Area} = \frac{\overset{175}{\cancel{350}}}{11} \times \frac{1}{\underset{9}{\cancel{18}}} = \frac{175}{99} = 1\tfrac{76}{99}\text{ sq. in.}$$

Ex. 597. Find the surface of a sphere if the diameter is (i.) 10 inches; (ii.) 1 foot 9 inches; (iii.) 2 feet 4 inches; (iv.) 7 feet; (v.) 4.2 feet; (vi.) 10.5 feet.

(i.) $S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times 5^2 = 100 \times \tfrac{22}{7} = \tfrac{2200}{7}$

$= 314\tfrac{2}{7}$ sq. in. $=$ 2 sq. ft. $26\tfrac{2}{7}$ sq. in.

(ii.) 1 ft. 9 in. $= 1\tfrac{3}{4}$ ft.

$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times (\tfrac{7}{8})^2 = \cancel{4} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\overset{7}{\cancel{49}}}{\underset{\underset{8}{\cancel{16}}}{\cancel{64}}} = \frac{77}{8}$$

$= 9\tfrac{5}{8}$ sq. ft. $=$ 9 sq. ft. 90 sq. in.

(iii.) 2 ft. 4 in. $= 2\tfrac{1}{3}$ ft.

$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times (\tfrac{7}{6})^2 = \overset{2}{\cancel{4}} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{7}}{\underset{3}{\cancel{6}}} \times \frac{7}{\underset{3}{\cancel{6}}} = \frac{154}{9}$$

$= 17\tfrac{1}{9}$ sq. ft. $=$ 17 sq. ft. 16 sq. in.

(iv.) $S = 4\pi R^2 = 4\times\frac{22}{7}\times\left(\frac{7}{2}\right)^2 = \overset{2}{\cancel{4}}\times\frac{22}{\cancel{7}}\times\frac{\cancel{7}}{\cancel{2}}\times\frac{7}{\cancel{2}} = 154$ sq. ft.

(v.) $S = 4\pi R^2 = 4\times\frac{22}{7}\times\left(\frac{2.1}{1}\right)^2 = 4\times\frac{22}{\cancel{7}}\times\overset{0.3}{\cancel{2.1}}\times 2.1$

$= 55.44$ sq. ft.

(vi.) $S = 4\pi R^2 = 4\times\frac{22}{7}\times(5.25)^2 = 4\times\frac{22}{\cancel{7}}\times\overset{0.75}{\cancel{5.25}}\times 5.25$

$= 346.5$ sq. ft.

Ex. 598. Find the diameter of a sphere if the surface is (i.) 616 square inches; (ii.) $38\frac{1}{2}$ square feet; (iii.) 9856 square feet.

$S = 4\pi R^2.$	$S = 4\pi R^2.$	$S = 4\pi R^2.$
$616 = 4\times\frac{22}{7}\times R^2.$	$38\frac{1}{2} = 4\times\frac{22}{7}\times R^2.$	$9856 = 4\times\frac{22}{7}\times R^2.$
$R^2 = \frac{\overset{\overset{7}{\cancel{28}}}{\cancel{616}}}{1}\times\frac{1}{\cancel{4}}\times\frac{7}{\cancel{22}}$	$R^2 = \frac{\overset{7}{\cancel{77}}}{2}\times\frac{1}{4}\times\frac{7}{\underset{2}{\cancel{22}}}$	$R^2 = \frac{\overset{\overset{112}{\cancel{448}}}{\cancel{9856}}}{1}\times\frac{1}{\cancel{4}}\times\frac{7}{\cancel{22}}$
$= 49.$	$= \frac{49}{16}.$	$= 784.$
$R = 7$ in.	$R = \frac{7}{4}$ ft.	$R = 28$ ft.
$D = 14$ in. $= 1$ ft. 2 in.	$D = \frac{7}{2}$ ft. $= 3$ ft. 6 in.	$D = 56$ ft.

Ex. 599. The circumference of a dome in the shape of a hemisphere is 66 feet; how many square feet of lead are required to cover it?

$$S = 2\pi R^2 = 66\,R.$$

$$R = \frac{66}{2\pi} = \frac{\overset{3}{\cancel{66}}}{2}\times\frac{7}{\cancel{22}} = \frac{21}{2}.$$

$$S = 66\times\tfrac{21}{2} = 693 \text{ sq. ft.}$$

Ex. 600. If the ball on the top of St. Paul's Cathedral in London is 6 feet in diameter, what would it cost to gild it at 7 cents per square inch?

$$S = 4\pi R^2 = 36\times\tfrac{22}{7} = \tfrac{792}{7}$$

$$= 113\tfrac{1}{7} \text{ sq. ft.} = 16{,}292\tfrac{4}{7} \text{ sq. in.}$$

$$\text{Cost} = 16{,}292\tfrac{4}{7}\times \$0.07 = \$1140.48.$$

Ex. 601. What is the numerical value of the radius of a sphere if its surface has the same numerical value as the circumference of a great circle?

$$S = 4\pi R^2.$$
$$C = 2\pi R.$$
$$4\pi R^2 = 2\pi R.$$
$$2R = 1,$$
$$R = \tfrac{1}{2}.$$

Ex. 602. Find the surface of a lune if its angle is 30°, and the total surface of the sphere is 4 square feet.

$$L : S = 30° : 360°.$$
$$L : 4 = 1 : 12.$$
$$L = \tfrac{4}{12} = \tfrac{1}{3} \text{ sq. ft.}$$

Ex. 603. What fractional part of the whole surface of a sphere is a spherical triangle whose angles are 43° 27′, 81° 57′, and 114° 36′?

$$E = 43°\,27' + 81°\,57' + 114°\,36' - 180° = 60°.$$
$$\frac{\Delta}{S} = \frac{E}{720} = \frac{60}{720} = \frac{1}{12}.$$

Ex. 604. The angles of a spherical triangle are 60°, 70°, and 80°. The radius of the sphere is 14 feet. Find the area of the triangle in square feet.

$$E = 60° + 70° + 80° - 180° = 30°.$$
$$S = 4\pi R^2 = 4 \times \frac{22}{\cancel{7}} \times \overset{2}{\cancel{14}} \times 14 = 2464 \text{ sq. ft.}$$
$$\frac{\Delta}{S} = \frac{E}{720}.$$
$$\frac{\Delta}{2464} = \frac{30}{720} = \frac{1}{24}.$$
$$\Delta = \tfrac{2464}{24} = 102\tfrac{2}{3} \text{ sq. ft.}$$

Ex. 605. The sides of a spherical triangle are 38°, 74°, and 128°. The radius of the sphere is 14 feet. Find the area of the polar triangle in square feet.

This exercise is impossible, since (§ 731) each side of a spherical Δ $<$ sum of the other two sides and $128° > 38° + 74°$.

The spherical angles of the base

Ex. 606. Find the area of a spherical polygon on a sphere whose radius is $10\frac{1}{2}$ feet, if its angles are 100°, 120°, 140°, and 160°.

$$E = T - (n-2)\,180^\circ.$$
$$T = 100^\circ + 120^\circ + 140^\circ + 160^\circ = 520^\circ.$$
$$E = 520^\circ - (2 \times 180^\circ)$$
$$= 520^\circ - 360^\circ = 160^\circ.$$
$$\frac{\text{Area}}{S} = \frac{160}{720} = \frac{2}{9}.$$
$$S = 4\pi R^2 = 4 \times \tfrac{22}{7} \times 10.5 \times 10.5 = 1386.$$
$$\frac{\text{Area}}{1386} = \frac{2}{9}.$$
$$\text{Area} = \frac{2 \times 1386}{9} = 308 \text{ sq. ft.}$$

Ex. 607. ~~The planes of the faces~~ of a quadrangular spherical pyramid ~~make~~ with each other ~~angles of~~ 80°, 100°, 120°, and 150°; and a length of a lateral edge of the pyramid is 42 feet. Find the area of its base in square feet.

$$E = T - (n-2)\,180^\circ.$$
$$T = 80^\circ + 100^\circ + 120^\circ + 150^\circ = 450^\circ.$$
$$E = 450^\circ - (2 \times 180^\circ)$$
$$= 450^\circ - 360^\circ = 90^\circ.$$
$$\frac{\text{Area}}{S} = \frac{90}{720} = \frac{1}{8}.$$
$$S = 4\pi R^2 = 4 \times \frac{22}{\cancel{7}} \times \overset{6}{\cancel{42}} \times 42 = 22{,}176.$$
$$\frac{\text{Area}}{22176} = \frac{1}{8}.$$
$$\text{Area} = \tfrac{1}{8} \text{ of } 22{,}176 = 2772 \text{ sq. ft.}$$

Ex. 608. The planes of the faces of a triangular spherical pyramid make with each other angles of 40°, 60°, and 100°, and the area of the base of the pyramid is 4π square feet. Find the radius of the sphere.

$$E = 40^\circ + 60^\circ + 100^\circ - 180^\circ = 20^\circ.$$
$$\frac{\Delta}{S} = \frac{E}{120} = \frac{20}{120} = \frac{1}{6}.$$
$$\frac{4\pi}{4\pi R^2} = \frac{1}{6}.$$
$$R^2 = 6,$$
$$R = \sqrt{6} = 2.72 \text{ ft.}$$

Ex. 609. The diameter of a sphere is 21 feet. Find the curved surface of a segment whose height is 5 feet.

$$\text{Area} = h \times 2\pi R = 5 \times \cancel{2} \times \frac{22}{\cancel{7}} \times \frac{\overset{3}{\cancel{21}}}{\cancel{2}} = 330 \text{ sq. ft.}$$

Ex. 610. What is the area of a zone of one base whose height is h, and the radius of the base r? What would be the area if the height were twice as great?

$$\text{Area} = \pi \times (\text{chord})^2 = \pi \times (\sqrt{h^2 + r^2})^2 = \pi(h^2 + r^2).$$

If the height were twice as great, $\text{area} = 2\pi(h^2 + r^2).$

Ex. 611. In a sphere whose radius is R, find the height of a zone whose area is equal to that of a great circle.

$$Z = h \times 2\pi R.$$

$$\text{Great circle} = \pi R^2.$$

$$2\pi Rh = \pi R^2.$$

$$2h = R.$$

$$h = \frac{R}{2}.$$

Ex. 612. The altitude of the torrid zone is about 3200 miles. Find its area in square miles, assuming the earth to be a sphere with a radius of 4000 miles.

$$Z = h \times 2\pi R = 3200 \times 2 \times \frac{22}{7} \times 4000 = 80{,}457{,}143 \text{ sq. mi.}$$

Ex. 613. A plane divides the surface of a sphere of radius R into two zones, such that the surface of the greater is a mean proportional between the entire surface and the surface of the smaller. Find the distance of the plane from the centre of the sphere.

$$\frac{2\pi R(2R - h)}{2\pi Rh} = \frac{2\pi Rh}{4\pi R^2}.$$

$$\frac{2R - h}{h} = \frac{h}{2R}.$$

$$4R^2 - 2Rh = h^2.$$

$$h^2 + 2Rh + R^2 = 5R^2.$$

$$h + R = \pm R\sqrt{5}.$$

$$h = -R \pm R\sqrt{5}.$$

$$\text{Distance from centre} = R\sqrt{5} - R - R = R(\sqrt{5} - 2).$$

Ex. 614. If a sphere of radius r is cut by two planes equally distant from the centre, so that the area of the zone comprised between the planes is equal to the sum of the areas of its bases, find the distance of either plane from the centre.

If a denotes the radius of the section,

$$a = \sqrt{r^2 - \tfrac{1}{4}h^2}.$$

$$Z = 2\pi rh.$$

$$2B = 2\pi a^2 = 2\pi(r^2 - \tfrac{1}{4}h^2).$$

$$\therefore\ 2\pi rh = 2\pi(r^2 - \tfrac{1}{4}h^2).$$

$$4\pi rh = 4\pi r^2 - \pi h^2.$$

$$h^2 + 4rh = 4r^2.$$

$$h^2 + 4rh + 4r^2 = 8r^2.$$

$$h + 2r = 2r\sqrt{2}.$$

$$h = 2r\sqrt{2} - 2r.$$

$$\frac{h}{2} = r\sqrt{2} - r = r(\sqrt{2} - 1).$$

Ex. 615. Find the area of the zone generated by an arc of 30°, of which the radius is r, and which turns around a diameter passing through one of its extremities.

Let AC denote the radius and AB the distance from the centre to the section.

Then

$$AB = \sqrt{r^2 - \left(\frac{r}{2}\right)^2} = \sqrt{\tfrac{3}{4}r^2} = \frac{r}{2}\sqrt{3}.$$

$$BC = r - \frac{r}{2}\sqrt{3}.$$

That is,

$$h = \frac{r}{2}(2 - \sqrt{3}).$$

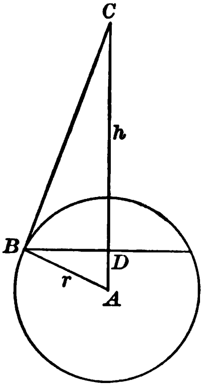
$$Z = 2\pi rh$$

$$= 2\pi r \times \frac{r}{2}(2 - \sqrt{3}) = \pi r^2(2 - \sqrt{3}).$$

Ex. 616. Find the area of the zone of a sphere of radius r, illuminated by a lamp placed at the distance h from the sphere.

In the similar $\triangle$ ADB and ABC,

$$\frac{AD}{AB}=\frac{AB}{AC},$$
$$\frac{AD}{r}=\frac{r}{r+h},$$
$$AD=\frac{r^2}{r+h},$$
$$DE=r-\frac{r^2}{r+h}$$
$$=\frac{rh}{r+h}.$$
$$Z=2\pi r\times\frac{rh}{r+h}$$
$$=\frac{2\pi r^2h}{r+h}.$$



Ex. 617. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

$$Z=\frac{2\pi r^2h}{r+h}. \qquad \text{(Ex. 616)}$$

But $$h=r.$$
$$Z=\frac{2\pi r^3}{2r}=\pi r^2.$$
$$S=4\pi r^2.$$
$$\frac{Z}{S}=\frac{\pi r^2}{4\pi r^2}=\frac{1}{4}.$$

Ex. 618. To what height must a man be raised above the earth in order that he may see one-sixth of its surface?

$$S=4\pi r^2,\text{ and } Z=\frac{S}{6}=\tfrac{2}{3}\pi r^2.$$

But $$Z=\frac{2\pi r^2h}{r+h}. \qquad \text{(Ex. 616)}$$
$$\therefore \frac{2\pi r^2h}{r+h}=\frac{2\pi r^2}{3}.$$

That is, $$\frac{h}{r+h}=\frac{1}{3}.$$
$$\therefore 3h=r+h.$$
$$2h=r,\quad h=\frac{r}{2}.$$

Ex. 619. Two cities are 200 miles apart. To what height must a man ascend from one city in order that he may see the other, supposing the circumference of the earth to be 25,000 miles?

$$\text{Area of } \odot = 200^2\pi = 40{,}000\pi.$$

But $$Z = \frac{25000 \times \frac{12500h}{\pi}}{\frac{12500}{\pi} + h}; \qquad \text{(Ex. 616)}$$

$$\therefore\ 40{,}000\pi = \frac{25000 \times \frac{12500h}{\pi}}{\frac{12500}{\pi} + h}. \qquad (\S\ 768)$$

$$2\pi = \frac{5 \times \frac{3125h}{\pi}}{\frac{12500}{\pi} + h}.$$

$$25{,}000\pi + 2\pi^2 h = 15{,}625\,h.$$

$$h = \frac{25000\pi}{15625 - 2\pi^2} = \frac{3850000}{765625 - 968}$$

$$= \frac{3850000}{764657} = 5.03 \text{ mi.}$$

Ex. 620. Find the volume of a sphere if the diameter is (i.) 13 inches; (ii.) 3 feet 6 inches; (iii.) 10 feet 6 inches; (iv.) 17 feet 6 inches; (v.) 14.7 feet; (vi.) 42 feet.

(i.) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times \frac{22}{7} \times \frac{13^3}{1} = \frac{1}{\underset{3}{\cancel{6}}} \times \frac{\overset{11}{\cancel{22}}}{7} \times \frac{2197}{1} = 1150\frac{17}{21}$ cu. in.

(ii.) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{2}} \times \frac{7}{2} \times \frac{7}{2} = \frac{539}{24} = 22\frac{11}{24}$ cu. ft.

(iii.) $V = \frac{1}{6}\pi D^3 = \frac{1}{\underset{2}{\cancel{6}}} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\overset{\cancel{3}}{\cancel{21}}}{2} \times \frac{21}{2} \times \frac{21}{2} = \frac{4851}{8} = 606\frac{3}{8}$ cu. ft.

(iv.) $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\overset{5}{\cancel{35}}}{\cancel{2}} \times \frac{35}{2} \times \frac{35}{2} = \frac{67375}{24} = 2807\frac{7}{24}$ cu. ft.

(v.) $V = \frac{1}{6}\pi D^3 = \frac{1}{\cancel{6}} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \overset{\overset{7}{\cancel{2.1}}}{\cancel{14.7}} \times 14.7 \times 14.7 = 1663.893$ cu. ft.

(vi.) $V = \frac{1}{6}\pi D^3 = \frac{1}{\cancel{6}} \times \frac{22}{\cancel{7}} \times \overset{\cancel{6}}{\cancel{42}} \times 42 \times 42 = 38{,}808$ cu. ft.

Ex. 621. Find the diameter of a sphere if the volume is (i.) 75 cubic feet 1377 cubic inches; (ii.) 179 cubic feet 1152 cubic inches; (iii.) 1047.816 cubic feet; (iv.) 38.808 cubic yards.

(i.) 75 cu. ft. 1377 cu. in. = 130,977 cu. in.

$$V = \frac{1}{6}\pi D^3.$$

$$130{,}977 = \frac{1}{6} \times \frac{22}{7} D^3.$$

$$D^3 = \overset{11907}{\cancel{130977}} \times \overset{3}{\cancel{6}} \times \frac{7}{\cancel{22}} = 250{,}047.$$

$$D = 63 \text{ in.} = 5 \text{ ft. } 3 \text{ in.}$$

(ii.) 179 cu. ft. 1152 cu. in. = 310,464 cu. in.

$$V = \frac{1}{6}\pi D^3.$$

$$310{,}464 = \frac{1}{6} \times \frac{22}{7} D^3.$$

$$D^3 = \overset{14112}{\cancel{310464}} \times 6 \times \frac{7}{\cancel{22}} = 592{,}704.$$

$$D = 84 \text{ in.} = 7 \text{ ft.}$$

(iii.) $V = \frac{1}{6}\pi D^3.$

$$1047.816 = \frac{1}{6} \times \frac{22}{7} D^3.$$

$$D^3 = \overset{47.628}{\cancel{1047.816}} \times 6 \times \frac{7}{\cancel{22}} = 2000.376.$$

$$D = 12.6 \text{ ft.}$$

(iv.) $V = \frac{1}{6}\pi D^3.$

$$38.808 = \frac{1}{6} \times \frac{22}{7} D^3.$$

$$D^3 = \overset{1.764}{\cancel{38.808}} \times 6 \times \frac{7}{\cancel{22}} = 84.088.$$

$$D = 4.2 \text{ yds.}$$

Ex. 622. Find the volume of a sphere whose circumference is 45 feet.

$$D = \frac{45}{\pi}.$$

$$V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times \pi \times \frac{45^3}{\pi^3} = \frac{1}{6} \times \frac{7}{22} \times \frac{7}{22} \times 45^3$$

$$= \frac{1488375}{968} = 1537.6 \text{ cu. ft.}$$

Ex. 623. Find the volume V of a sphere in terms of the circumference C of a great circle.

$$V = \tfrac{1}{6}\pi D^3.$$

$$C = \pi D.$$

$$D = \frac{C}{\pi}.$$

$$V = \frac{\pi}{6} \times \frac{C^3}{\pi^3} = \frac{C^3}{6\pi^2}.$$

Ex. 624. Find the radius R of a sphere, having given the volume V.

$$V = \tfrac{4}{3}\pi R^3.$$

$$R^3 = \frac{3V}{4\pi}.$$

$$R = \sqrt[3]{\frac{3V}{4\pi}}.$$

Ex. 625. Find the radius R of a sphere, if its circumference and its volume have the same numerical value.

$$\tfrac{4}{3}\pi R^3 = 2\pi R.$$

$$\tfrac{2}{3}R^2 = 1,$$

$$2R^2 = 3,$$

$$R^2 = 1.5,$$

$$R = \sqrt{1.5} = \tfrac{1}{2}\sqrt{6} = 1.2247.$$

Ex. 626. If an iron ball 4 inches in diameter weighs 9 pounds, what is the weight of a hollow iron shell 2 inches thick, whose external diameter is 20 inches?

$$V : V'' = 20^3 : 4^3 = 8000 : 64 = 125 : 1.$$

$$V \text{ weighs } 125 \times 9 \text{ lbs.} = 1125 \text{ lbs.}$$

$$V' : V'' = 16^3 : 4^3 = 4^6 : 4^3 = 4^3 : 1 = 64 : 1.$$

$$V' \text{ weighs } 64 \times 9 \text{ lbs.} = 576 \text{ lbs.}$$

$$V - V' = 549 \text{ lbs.}$$

Ex. 627. The radius of a sphere is 7 feet; what is the volume of a spherical wedge whose angle is 36°?

$$\frac{L}{S} = \frac{36^\circ}{360^\circ} = \frac{1}{10}.$$

$$S = 4\pi R^2 = 4 \times \frac{22}{\cancel{7}} \times \cancel{7} \times 7 = 616 \text{ sq. ft.}$$

$$\frac{L}{616}=\frac{1}{10}.$$

$$L=61\tfrac{3}{5}.$$

$$V=\tfrac{1}{3}RL=\tfrac{1}{3}\times 7\times \tfrac{308}{5}$$

$$=\tfrac{2156}{15}=143\tfrac{11}{15}\text{ cu. ft.}$$

Ex. 628. What is the angle of a spherical wedge, if its volume is 1 cubic foot, and the volume of the entire sphere is 6 cubic feet?

$$\frac{L}{S}=\frac{\text{Angle of Lune}}{360^\circ}.$$

$$\frac{1}{6}=\frac{\text{Angle}}{360^\circ}.$$

$$\text{Angle}=60^\circ.$$

Ex. 629. What is the volume of a spherical sector, if the area of the zone which forms its base is 3 square feet, and the radius of the sphere is 1 foot?

$$V=\tfrac{1}{3}RZ=\tfrac{1}{3}\times 1\times 3=1\text{ cu. ft.}$$

Ex. 630. The radius of the base of the segment of a sphere is 16 inches, and the radius of the sphere is 20 inches; find its volume.

$$V=\tfrac{1}{2}\pi r^2a+\tfrac{1}{6}\pi a^3=\frac{1}{\cancel{2}}\times\frac{\overset{11}{\cancel{22}}}{7}\times\frac{256}{1}\times\frac{8}{1}+\frac{1}{\underset{3}{\cancel{6}}}\times\frac{\overset{11}{\cancel{22}}}{7}\times\frac{512}{1}$$

$$=\frac{22528}{7}+\frac{5632}{21}=3218\tfrac{2}{7}+268\tfrac{4}{21}$$

$$=3486\tfrac{10}{21}\text{ cu. in.}=2\text{ cu. ft. }30\tfrac{10}{21}\text{ cu. in.}$$

Ex. 631. The inside of a wash-basin is in the shape of the segment of a sphere; the distance across the top is 16 inches, and its greatest depth is 6 inches; find how many pints of water it will hold, reckoning $7\tfrac{1}{2}$ gallons to the cubic foot.

$$V=\tfrac{1}{2}\pi r^2a+\tfrac{1}{6}\pi a^3.\quad a=6.\quad r=8.$$

$$V=\frac{1}{\cancel{2}}\times\frac{\overset{11}{\cancel{22}}}{7}\times 64\times 6+\frac{1}{\cancel{6}}\times\frac{22}{7}\times\overset{36}{\cancel{216}}=\frac{4224}{7}+\frac{792}{7}$$

$$=\frac{5016}{7}\text{ cu. in.}=\frac{\overset{209}{\cancel{5016}}}{7}\times\frac{1}{\underset{\underset{\underset{3}{\cancel{9}}}{\cancel{72}}}{\cancel{1728}}}\times\frac{\overset{5}{\cancel{15}}}{2}\times\cancel{8}$$

$$=\frac{1045}{42}=24\tfrac{37}{42}\text{ pts.}$$

Ex. 632. What is the height of a zone, if its area is S, and the volume of the sphere to which it belongs is V?

$$V = \tfrac{4}{3}\pi R^3.$$

$$S = 2\pi Rh.$$

$$h = \frac{S}{2\pi R}.$$

$$V = \tfrac{4}{3}\pi R^3.$$

$$R = \sqrt[3]{\frac{3V}{4\pi}}.$$

$$2\pi R = 2\pi\sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{6\pi^2 V}.$$

$$\therefore\ h = \frac{S}{\sqrt[3]{6\pi^2 V}}.$$

Ex. 633. The radii of the bases of a spherical segment are 6 feet and 8 feet, and its height is 3 feet; find its volume.

$$V = \frac{h}{2}(\pi r^2 + \pi r'^2) + \frac{\pi h^3}{6} = \tfrac{3}{2}(\pi 6^2 + \pi 8^2) + \frac{\pi 3^3}{6} \qquad (\S\ 790)$$

$$= 150\pi + \frac{9\pi}{2} = \frac{309}{\cancel{2}} \times \frac{\overset{11}{\cancel{22}}}{7} = \frac{3399}{7} = 485\tfrac{4}{7} \text{ cu. ft.}$$

Ex. 634. Find the volume of a triangular spherical pyramid if the angles of the spherical triangle which forms its base are each 100°, and the radius of the sphere is 7 feet.

$$E = 3 \times 100° - 180° = 120°.$$

$$S = 4\pi R^2 = 4 \times \frac{22}{\cancel{7}} \times \overset{7}{\cancel{49}} = 616.$$

$$\frac{\Delta}{S} = \frac{E}{720}.$$

$$\frac{\Delta}{616} = \frac{120}{720} = \frac{1}{6}.$$

$$\Delta = \frac{308}{3} \text{ sq. ft.}$$

$$V = \frac{1}{3} \times 7 \times \frac{308}{3} = \frac{2159}{9} = 239\tfrac{8}{9} \text{ cu. ft.}$$

Ex. 635. The circumference of a sphere is 28 π feet; find the volume of that part of the sphere included by the faces of a trihedral angle at the centre, the dihedral angles of which are 80°, 105°, and 140°.

$$E = 80° + 105° + 140° - 180° = 145°.$$

$$S = 4\pi R^2 = 4 \times \frac{22}{\cancel{7}} \times \overset{28}{\cancel{196}} = 2464.$$

$$\frac{\Delta}{S} = \frac{E}{720}.$$

$$\frac{\Delta}{2464} = \frac{145}{720}.$$

$$\Delta = \overset{154}{\cancel{2464}} \times \frac{\overset{29}{\cancel{145}}}{\underset{\underset{9}{\cancel{45}}}{\cancel{720}}} = \frac{4466}{9}.$$

$$V = \frac{1}{3} \times \frac{14}{1} \times \frac{4466}{9} = \frac{62524}{27} = 2315\tfrac{19}{27} \text{ cu. ft.}$$

Ex. 636. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°, and a lateral edge of the pyramid is 3½ feet; find the volume of the pyramid.

$$E = 80° + 100° + 120° + 150° - 360° = 90°.$$

$$S = 4\pi R^2 = \cancel{4} \times \frac{22}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{2}} \times \frac{7}{\cancel{2}} = 154 \text{ sq. ft.}$$

$$\frac{\Delta}{S} = \frac{E}{720}.$$

$$\frac{\Delta}{154} = \frac{90}{720} = \frac{1}{8}.$$

$$\Delta = \frac{154}{8} = \frac{77}{4} = 19\tfrac{1}{4} \text{ sq. ft.}$$

$$V = \frac{1}{3} \times \frac{7}{2} \times \frac{77}{4} = \frac{539}{24} = 22\tfrac{11}{24} \text{ cu. ft.}$$

Ex. 637. The radius of the base of the segment of a sphere is 40 feet, and its height is 20 feet; find its volume.

$$V = \tfrac{1}{2}\pi r^2 a + \tfrac{1}{6}\pi a^3 = \frac{1}{\cancel{2}} \times \frac{22}{7} \times \frac{1600}{1} \times \overset{10}{\cancel{20}} + \frac{1}{\underset{3}{\cancel{6}}} \times \frac{22}{7} \times \overset{4000}{\cancel{8000}}$$

$$= \frac{352000}{7} + \frac{88000}{21} = \frac{1144000}{21} = 54{,}476\tfrac{4}{21} \text{ cu. ft.}$$

Ex. 638. Having given the volume V, and the height h, of a spherical segment of one base, find the radius r of the sphere.

$$V = \pi h^2\left(r - \frac{h}{3}\right) = \pi h^2 r - \tfrac{1}{3}\pi h^3.$$

$$r = \frac{V + \frac{1}{3}\pi h^3}{\pi h^2} = \frac{V}{\pi h^2} + \frac{h}{3}.$$

Ex. 639. Find the weight of a sphere of radius r, which floats in a liquid of specific gravity s, with one-fourth of its surface above the surface of the liquid. *The weight of a floating body is equal to the weight of the liquid displaced.*

$$Z = 2\pi rh.$$

$$\tfrac{1}{4}S = \pi r^2 = 2\pi rh.$$

$$h = \frac{r}{2}.$$

The volume out of the liquid $= \pi h^2\left(r - \frac{h}{3}\right) = \pi\frac{r^2}{4}\left(r - \frac{r}{6}\right)$

$$= \frac{\pi r^2}{4} \times \frac{5r}{6} = \frac{5\pi r^3}{24}.$$

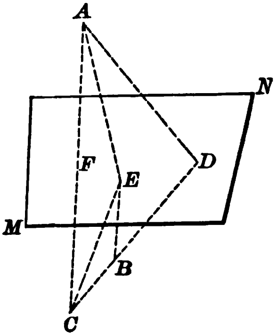
Volume of sphere $= \frac{4}{3}\pi r^3$.

Therefore the volume of the liquid displaced

$$= \tfrac{4}{3}\pi r^3 - \tfrac{5}{24}\pi r^3 = \tfrac{9}{8}\pi r^3.$$

This weighs $\frac{9}{8}\pi r^3 s$.

Ex. 640. Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of the plane shall be a maximum.



Let A and B be any two points on opposite sides of the plane MN; it is required to find the point in the plane MN the difference of whose distances from A and B shall be a maximum.

CONSTRUCTION. From A draw AF perpendicular to the plane MN, and prolong it, making $FC = AF$. Through C and B draw a line meeting the plane in some point, as D; then D is the required point.

PROOF. Let E be any other point of the plane MN. Draw EA, EC, EB, and AD; then, by § 482,

$$DC = DA, \quad EC = EA. \qquad (1)$$

$$\therefore\ DA - DB = DC - DB = BC.$$

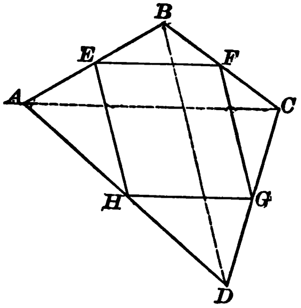
Also $$EA - EB = EC - EB < BC. \qquad (§\ 137)$$

Hence $$DA - DB > EA - EB.$$

DISCUSSION. If the points A and B are equally distant from the plane, but not in the same perpendicular, CB will be parallel to the plane, and no point is determined. In this case there is no maximum difference of distances.

If the points A and B are equally distant from the plane and in the same perpendicular, CB is not determined in position. In this case the difference of distances is the constant zero.

Ex. 641. In any warped quadrilateral, that is, one whose sides do not all lie in the same plane, the middle points of the sides are the vertices of a parallelogram.



Let $ABCD$ be any quadrilateral, all of whose sides do not lie in the same plane; it is required to prove that the figure $EFGH$, formed by joining the middle points of the sides of $ABCD$, is a parallelogram.

PROOF. Draw the diagonals BD and AC. Now in the $\triangle\ ABD$, EH is $\parallel$ to BD (§ 312).

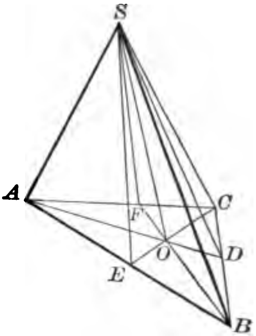
Also in triangle CBD, FG is $\parallel$ to BD.

Hence EH and FG are parallel (§ 485).

For like reason EF and HG are parallel.

Hence the opposite sides of $EFGH$ are $\parallel$, and the figure is a ▱.

Ex. 642. In any trihedral angle, the three planes bisecting the three dihedral angles intersect in the same straight line.



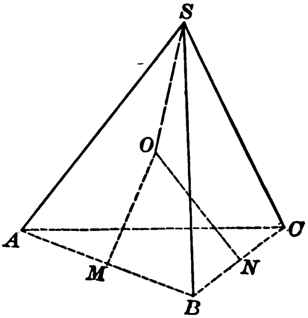
Let *S-ABC* be any trihedral ∠, and let the planes *SAD*, *SBF*, and *SCE* bisect the dihedrals *SA*, *SB*, and *SC*, respectively. We are to prove that these planes intersect in the same straight line.

PROOF. Let *SAD* and *SBF* intersect in the straight line *SO*; then every point in this line is equally distant from the planes *SAC* and *SAB* (§ 525).

For like reason, every point in *SO* is equally distant from the planes *SBA* and *SBC*. Hence every point in *SO* is equally distant from the planes *SCB* and *SCA*; therefore *SO* is in the bisector of the dihedral *SC*.

Hence the bisecting planes intersect in the same straight line *SO*.

Ex. 643. To draw a line through the vertex of any trihedral angle, making equal angles with its edges.



Let *S-ABC* be any trihedral ∠. It is required to draw a line through *S*, making equal ∠s with the edges *SA*, *SB*, and *SC*.

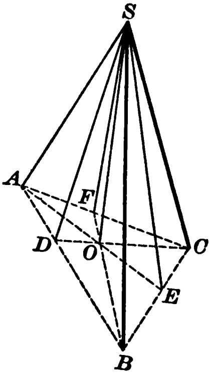
CONSTRUCTION. Take *SA* = *SB* = *SC*, and draw the lines *AB*, *BC*, and *AC*. In the plane *ABC* draw the ⊥s *MO* and *NO* to the sides *AB* and *BC* at their middle points respectively. Connect their common point *O* with the vertex *S*; then is *OS* the line required.

PROOF. The point *O* is equally distant from the three points *A*, *B*, and *C* (§ 122).

Hence *O* is the foot of the ⊥ from the point *S* to the plane *ABC* (§ 479).

Whence the ∠s *OSA*, *OSB*, and *OSC* are equal (§ 479).

Ex. 644. In any trihedral angle, the three planes passed through the edges and the respective bisectors of the opposite face angles intersect in the same straight line.



Let *S-ABC* be any trihedral ∠, and *SD*, *SE*, and *SF* be the bisectors of its face ∠s; we are to prove that the planes *ASE*, *BSF*, and *CSD* will intersect in the same straight line.

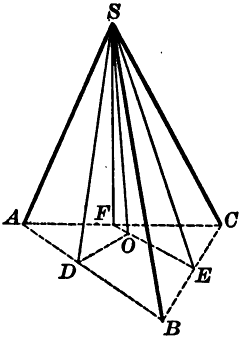
Proof. Take $SA = SB = SC$, and draw the lines *AB*, *BC*, and *CA*. Draw also *AE*, *BF*, and *CD*, *D*, *E*, and *F* being the points in which *SD*, *SE*, and *SF* cut *AB*, *BC*, and *CA*, respectively.

By construction the △s *SAB*, *SBC*, and *SCA* are isosceles; hence the bisectors *SD*, *SE*, and *SF* are medial lines of the respective △s (Ex. 14).

Hence *AE*, *BF*, and *CD* are the medial lines of the △ *ABC*.

These medial lines intersect in some point, as *O*. Hence the planes *ASE*, *BSF*, and *CSD* have two points *O* and *S* in common, and therefore intersect in the straight line *SO*.

Ex. 645. In any trihedral angle, the three planes passed through the bisectors of the face angles, and perpendicular to these faces respectively, intersect in the same straight line.

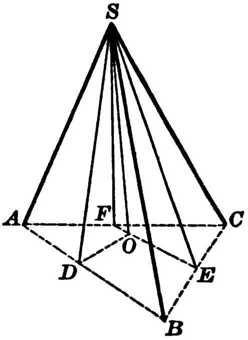


Let *S-ABC* be any trihedral ∠, and *SD*, *SE*, and *SF* the bisectors of its face ∠s; we are to prove that the planes through *SD*, *SE*, and *SF*, and ⊥ respectively to the faces *ASB*, *BSC*, and *CSA*, intersect in the same straight line.

Proof. Take $SA = SB = SC$, and draw the lines *AB*, *BC*, and *CA*. Let *D*, *E*, and *F* be their points of intersection with *SD*, *SE*, and *SF* respectively. In the △ *ABC*, at the points *D*, *E*, and *F*, draw ⊥s to the sides *AB*, *BC*, and *CA* respectively.

By construction, the △s *SAB*, *SBC*, and *SCA* are isosceles; hence the bisectors *SD*, *SE*, and *SF* are ⊥ to *AB*, *BC*, and *CA*, respectively, at their middle points.

Hence the ⊥s at D, E, and F, in the plane ABC, intersect in some point, as O (Ex. 31).



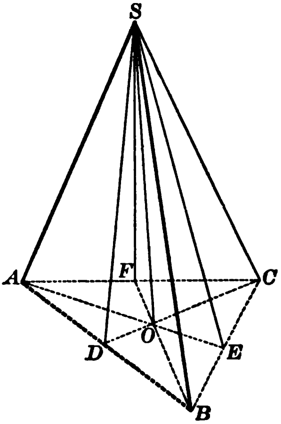
Since AB is ⊥ to DS and DO, it is ⊥ to the plane SDO (§ 472).

Hence the plane ABS is ⊥ to the plane SDO (§ 518).

For like reason the plane OES is ⊥ to the plane BSC, and the plane OFS to the plane CAS.

Hence the three planes passed through the bisectors of the face ∠s of the trihedral ∠, and ⊥ to these faces respectively, have the two points O and S in common, and therefore intersect in the straight line OS.

Ex. 646. In any trihedral angle, the three planes passed through the edges, perpendicular to the opposite faces respectively, intersect in the same straight line.



Let S-ABC be any trihedral ∠. We are to prove that the three planes passed through the edges SA, SB, and SC, ⊥ to the opposite faces SBC, SCA, and SAB, respectively, intersect in the same straight line.

Proof. Let the ⊥ planes through SC and SB intersect in the line SO. Through SA and SO pass a plane cutting the plane SBC in the line SE.

Through O, any point in SO, pass a plane ⊥ to SO. Let AB, BC, and AC be its intersections with the faces of the trihedral ∠, and AE, BF, and CD its intersections with the planes through the lateral edges. Each of the planes SAE, SBF, and SCD is ⊥ to the plane ABC (§ 518).

By hypothesis SCD is ⊥ to SAB. Hence AB is ⊥ to the plane SDC (a plane ⊥ to each of two other planes is ⊥ to their intersection).

Therefore CD is ⊥ to AB (§ 462).

For like reason BF is $\perp$ to AC. Hence in the $\triangle ABC$, BF and CD are lines drawn from the vertices B and C $\perp$ to the opposite sides AC and AB respectively; hence AE, which is a line from the vertex A through the intersection of the two $\perp$s BF and CD, is $\perp$ to the side BC (Ex. 32).

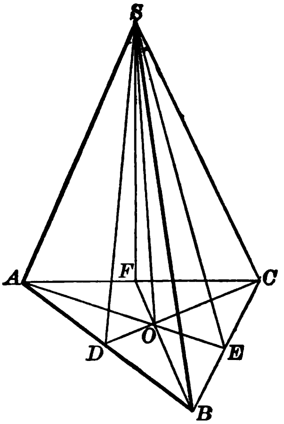
Since the plane SAE is $\perp$ to the plane ABC, and BE is $\perp$ to AE, BE is $\perp$ to the plane AES (§ 515).

Hence the plane BSC is $\perp$ to the plane ASE, or ASE is a plane through SA $\perp$ to the opposite face SBC (§ 518).

Hence the three planes through the edges SA, SB, and SC, $\perp$ to the opposite faces respectively, intersect in the straight line SO.

Ex. 647. In a tetrahedron, the planes passed through the three lateral edges and the middle points of the sides of the base intersect in a straight line.

Let S-ABC be any tetrahedron, and D, E, and F the middle points of the sides of its base; we are to prove that the planes SAE, SBF, and SCD intersect in the same straight line.

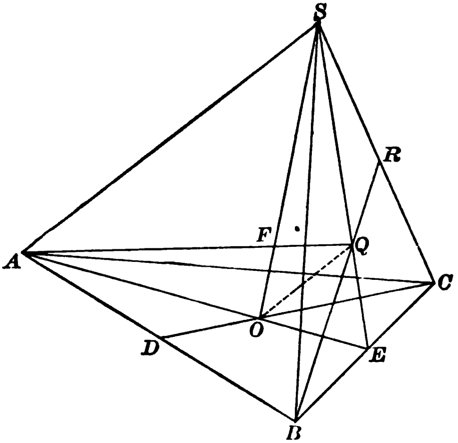


Proof. The lines AE, BF, and CD, being medial lines of the $\triangle ABC$, must intersect in some point, as O (Ex. 33).

Hence the three planes SAE, SBF, and SCD have two points S and O in common, and therefore intersect in the straight line SO.

Ex. 648. The lines joining each vertex of a tetrahedron with the point of intersection of the medial lines of the opposite face all meet in a point, called the *centre of gravity*, which divides each line so that the shorter segment is to the whole line in the ratio 1 : 4.

Let S-ABC be any tetrahedron, and AQ and SO the lines drawn from A and S to the intersections Q and O of the medial lines of the opposite faces BSC and ABC respectively; then AQ and SO intersect, and divide each other in the ratio 1 : 4.



Proof. Draw OQ. The lines AQ and SO are drawn in the $\triangle$ AES from the vertices A and S to the opposite sides; hence they intersect in some point, as F.

Now $EO = \frac{1}{3} EA$, and $EQ = \frac{1}{3} ES$. (Ex. 33)

Hence the $\triangle$ OEQ and AES are similar. (§ 326)

Therefore $OQ = \frac{1}{3} AS$,

and OQ and AS are parallel. (§ 312)

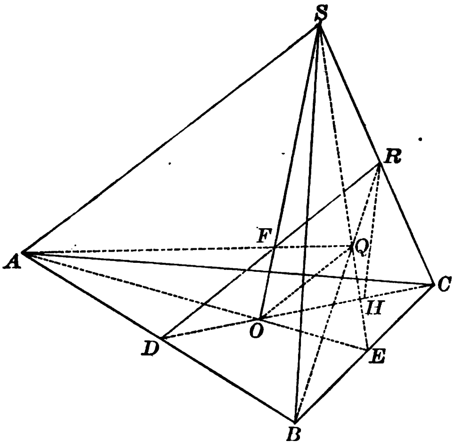
Hence the $\triangle$ OFQ and AFS are mutually equiangular and therefore similar. $\therefore OF = \frac{1}{3} FS$, and $QF = \frac{1}{3} FA$;

that is, $OF = \frac{1}{4} OS$, and $QF = \frac{1}{4} QA$.

In like manner it can be shown that the lines drawn from B and C to the intersection of the medial lines of the opposite faces pass through F and are divided at that point so that the shorter segment is to the whole line in the ratio 1 : 4.

Ex. 649. The straight lines joining the middle points of the opposite edges of a tetrahedron all pass through the centre of gravity of the tetrahedron, and are bisected by the centre of gravity.

Let S-ABC be any tetrahedron in which DR connects the middle points of the opposite edges AB and SC; we are to prove that DR is bisected by the centre of gravity of the tetrahedron.



Proof. Let O be the intersection of the medial lines AE and CD of the face ABC. Draw SO, and through R draw $RH \parallel$ to SO.

Since SO and RD lie in the plane DCS they evidently intersect in some point, as F. We are to prove that

$$OF = \tfrac{1}{4} OS, \text{ and } DF = FR.$$

Now $\quad HC = OH, \text{ and } RH = \tfrac{1}{2} SO. \quad$ (§ 309)

But $\quad DO = \tfrac{1}{2} OC. \quad$ (Ex. 33)

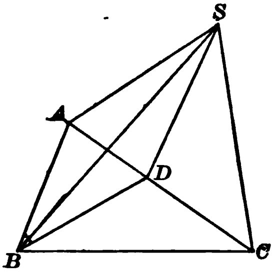
Hence $\quad DO = OH.$

Whence $\quad OF = \tfrac{1}{2} HR = \tfrac{1}{4} SO, \text{ and } DF = FR.$

($\triangle$s OFD and HRD similar, and $OD = \tfrac{1}{2} DH$.)

Hence $OF = \tfrac{1}{4} OS$, and $DF = FR$; that is, F is the centre of gravity of the tetrahedron and bisects DR. In like manner we could prove that the line joining the middle points of any other two opposite edges is bisected by the centre of gravity.

Ex. 650. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments which are proportional to the areas of the faces, including the dihedral angle.



Let the plane *SBD* bisect the dihedral ∠ *A-SB-C* of the trihedral *S-ABC*; we are to prove that

$$\frac{AD}{DC}=\frac{SAB}{SBC}.$$

PROOF. Since the tetrahedrons *S-ABD* and *S-DBC* have the same altitude, they are to each other as their bases; that is,

$$\frac{S\text{-}ABD}{S\text{-}DBC}=\frac{DBA}{DBC}. \qquad (1)$$

Since the ▲ *DBA* and *DBC* have the same altitude, they are to each other as their bases; that is,

$$\frac{DBA}{DBC}=\frac{DA}{DC}. \qquad (2)$$

From (1) and (2) we have

$$\frac{S\text{-}ABD}{S\text{-}DBC}=\frac{DA}{DC}. \qquad (3)$$

Consider *D* as the common vertex of the tetrahedrons *S-ABD* and *S-DBC*, and *SAB* and *SBC* as their bases respectively; then their altitudes are equal. (*D* is in the plane bisecting the dihedral ∠ *A-BS-C*, and every point in this plane is equally distant from the faces *SAB* and *SBC*.)

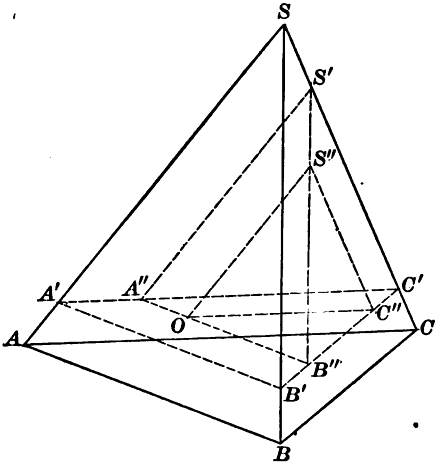
Hence $$\frac{D\text{-}SAB}{D\text{-}SBC}=\frac{SAB}{SBC}, \text{ or } \frac{S\text{-}ABD}{S\text{-}DBC}=\frac{SAB}{SBC}. \qquad (4)$$

From (3) and (4) we have $\frac{DA}{DC}=\frac{SAB}{SBC}.$

Ex. 651. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars let fall from any point within it upon the four faces.

Let O be any point within the tetrahedron $S\text{-}ABC$; we are to prove that the sum of the four $\perp$s from O upon the faces of the tetrahedron is equal to the altitude of the tetrahedron.

Proof. Through O pass the plane $A'B'C' \parallel$ to the face ABC; then will $S\text{-}A'B'C'$ be another regular tetrahedron (its faces will all be equiangular $\triangle$s). The altitude of $S\text{-}ABC$ is equal to that of $S\text{-}A'B'C'$ plus the $\perp$ from O upon the face ABC.



Through O pass the plane $S'A''B'' \parallel$ to the face SAB; then will $S'\text{-}A''B''C'$, or $C'\text{-}S'A''B''$, be a regular tetrahedron.

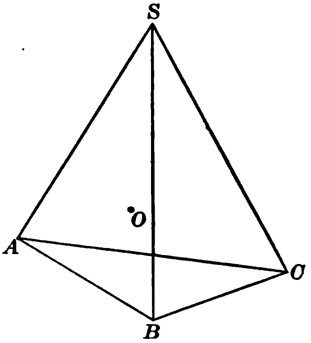
The altitude of $C'\text{-}SA'B'$ is equal to that of $C'\text{-}S'A''B''$ plus the $\perp$ from O upon the face SAB.

Again through O pass the plane $OS''C'' \parallel$ to the face SAC; then will $B''\text{-}OS''C''$ be a regular tetrahedron.

The altitude of $B''\text{-}A''S'C'$ is equal to that of $B''\text{-}OS''C''$ plus the $\perp$ from O upon the face SAC, and the altitude of $B''\text{-}OS''C''$ is equal to the $\perp$ from O upon the face SBC.

Hence the altitude of $S\text{-}ABC$ equals the sum of the four $\perp$s from O upon the four faces of the tetrahedron.

Ex. 652. Within a given tetrahedron, to find a point such that the planes passed through this point and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.



Let $S\text{-}ABC$ be any tetrahedron; it is required to find a point such that the planes passed through it and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.

CONSTRUCTION. Construct a plane ∥ to ABC at a distance equal to one-fourth the distance of S from ABC. Construct a second plane ∥ to SAB at a distance equal to one-fourth the distance of C from SAB. Construct a third plane ∥ to SAC at a distance equal to one-fourth the distance of B from SAC. Let O be the intersection of these three planes; then will O be the point required.

PROOF. By construction this point O will be within the tetrahedron. The pyramid having ABC as base and O as vertex is one-fourth $S\text{-}ABC$, since it has the same base and one-fourth the altitude of $S\text{-}ABC$.

For like reason $O\text{-}SAB = \frac{1}{4}\, C\text{-}SAB = \frac{1}{4}\, S\text{-}ABC$,

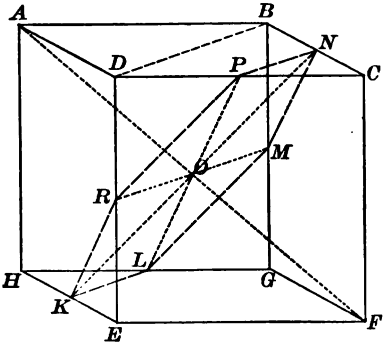
and $O\text{-}SAC = \frac{1}{4}\, B\text{-}SAC = \frac{1}{4}\, S\text{-}ABC$.

Now $O\text{-}SBC = \frac{1}{4}\, S\text{-}ABC$.

(It is the remainder after three-fourths are taken away.)

Hence O is the required point.

Ex. 653. To cut a cube by a plane so that the section shall be a regular hexagon.



Let $A\text{-}HEFG$ be any cube; it is required to cut it with a plane so that the section shall be a regular hexagon.

CONSTRUCTION. Draw the diagonal of the cube AF, and through its middle point O pass a plane ⊥ to it, forming the section $KLMNPR$; then will this section be a regular hexagon.

PROOF. The point P is equally distant from A and F (§ 482).

But $\overline{AP}^2 = \overline{AD}^2 + \overline{DP}^2$, and $\overline{FP}^2 = \overline{FC}^2 + \overline{CP}^2$.

Hence $\overline{DP}^2 = \overline{CP}^2$, or $DP = CP$.

That is, P is the middle point of the edge DC.

For like reason N is the middle point of BC, etc.

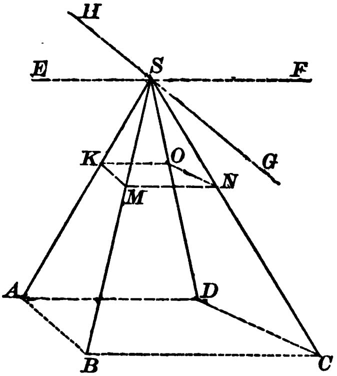
Hence $PN = \frac{1}{2} DB$, or each side of the section is equal to one-half the diagonal of a face (▲ CPN and CDB are similar, and $CN = \frac{1}{2} CB$).

The section through O, ∥ to $DAHE$, is equal to $DAHE$ and passes through P and L.

Hence $OP = OL = \frac{1}{2}$ diagonal DH.

Hence the ▲ OPR, ORK, etc., are equilateral ▲, and therefore the ∠s RPN, PNM, etc., are each equal to 120°. Hence the section is a regular hexagon.

Ex. 654. To cut a tetrahedral angle so that the section shall be a parallelogram.



Let S-$ABCD$ be any tetrahedral ∠; it is required to cut it by a plane so that the section made shall be a ▱.

CONSTRUCTION. Let ESF be the intersection of the planes of the opposite faces BSC and ASD, and HSG that of the planes of the opposite faces ASB and CSD.

These two lines of intersection determine a plane HSE through the vertex S.

Through K, any point in the edge SA, pass a plane ∥ to the plane HSE forming the section $KMNO$; then will $KMNO$ be a ▱.

PROOF. The lines KM and HG are parallel, since they are the intersections of the plane ASB with the ∥ planes $KMNO$ and ESH. For like reason ON and HG are parallel. Hence KM and ON are parallel (§ 111).

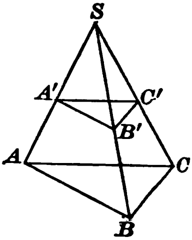
Similarly, we may prove that MN and KO are parallel.

Hence by definition $KMNO$ is a ▱.

DISCUSSION. The cutting plane may be made to pass through any point on the side of the plane ESH on which the tetrahedral angle lies. Thus there is an indefinite number of solutions of the problem.

Ex. 655. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.

Let $A'B'C'$ be a section ∥ to the face ABC of the tetrahedron S-ABC; we are to prove that S-$A'B'C'$ is similar to S-ABC.



PROOF. $A'B'$, $B'C'$, $C'A'$ are ∥ respectively to AB, BC, CA (§ 492).

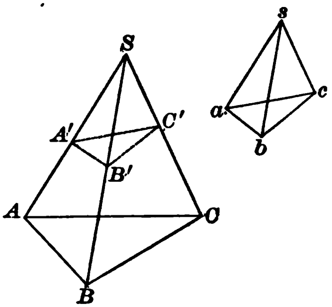
Hence the faces $SA'B'$, $SB'C'$, and $SC'A'$ are similar respectively to SAB, SBC, and SCA (§ 321).

The trihedral ∠ S is common to the two tetrahedrons, and the trihedral ∠s A', B', and C' are equal to A, B, and C respectively, since trihedral ∠s having their face ∠s equal, each to each, and similarly arranged are equal (§ 541).

Since the tetrahedrons have their solid ∠s equal, and their faces similar and similarly arranged, they are themselves similar (§ 614).

Ex. 656. Two tetrahedrons, having a dihedral angle of one equal to a dihedral angle of the other, and the faces including these dihedral angles respectively similar, and similarly placed, are similar.

Let S-ABC and s-abc be two tetrahedrons having the dihedral ∠ SB equal to the dihedral ∠ sb, and the faces SAB and SBC similar respectively to sab and sbc; we are to prove that the two tetrahedrons are similar.



PROOF. Place the tetrahedron s-abc on S-ABC so that their vertices and equal dihedral ∠s coincide. Then, from the equality of the face ∠s, sc will fall on SC, and sa on SA.

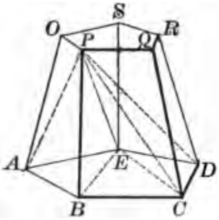
Let S-$A'B'C'$ denote the position of s-abc; then $A'B'$ and $B'C'$ will be ∥ to AB and BC respectively, since the faces SAB and SBC are similar to sab and sbc respectively.

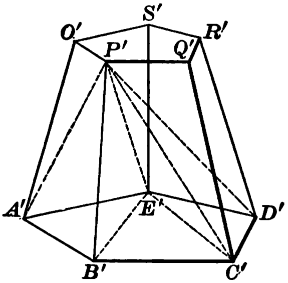
Hence $A'B'C'$ is a section ∥ to the face ABC (§ 498).

Therefore, by Ex. 655, S-$A'B'C'$, or s-abc, is similar to S-ABC.

Ex. 657. Two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed, are similar.

Let OD and $O'D'$ be two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed; we are to prove that they are similar.





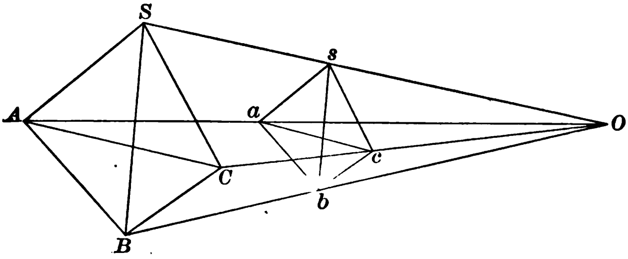
Proof. Any two corresponding solid ∠s, as B and B', are made up of equal solid ∠s of the similar tetrahedrons similarly placed; hence both the solid ∠s and the face ∠s are equal respectively.

Moreover, from the similar tetrahedrons we have

$$\frac{AB}{A'B'}=\frac{PB}{P'B'}=\frac{BC}{B'C'}=\text{etc.};$$

that is, the homologous edges are proportional. Hence the corresponding solid ∠s being equal, and the corresponding faces similar, the polyhedrons are similar (§ 614).

Ex. 658. If the homologous faces of two similar pyramids are respectively parallel, the straight lines which join the homologous vertices of the pyramids meet in a point.

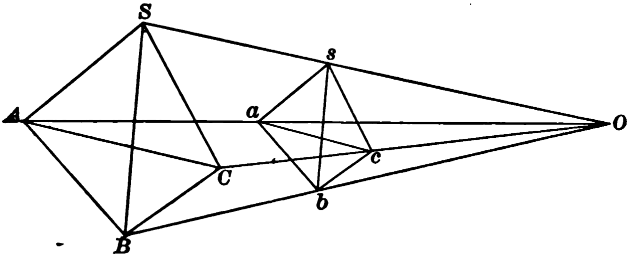


Let the homologous faces of the two similar pyramids S-ABC and s-abc be $\parallel$ respectively; we are to prove that the lines Ss, Aa, Bb, and Cc through the homologous vertices will meet in some point, as O.

PROOF. Since the homologous faces are parallel, the homologous edges are also parallel. Let O be the point in which Aa meets Ss; then the △ OSA and Osa, being mutually equiangular, are similar.

Hence $$OS : Os :: AS : as.$$

$$\therefore\ OS - Os\,[= Ss] : SO :: AS - as : AS. \qquad (1)$$



Again, let O' be the point in which Cc cuts Ss produced; then

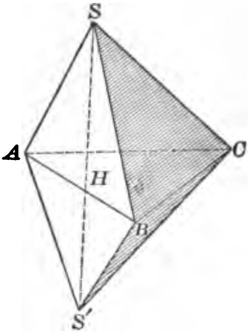
$$O'S : O's :: CS : cs :: AS : as.$$

$$\therefore\ Ss : SO' :: AS - as : AS. \qquad (2)$$

From (1) and (2) we obtain $SO' = SO$; that is, Cc cuts Ss in the same point that Aa does. In like manner it may be shown that Bb cuts Cc in the same point that Ss does. Hence all the lines Ss, Aa, Bb, and Cc meet in a point.

Ex. 659. Two symmetrical tetrahedrons are equivalent.

Let S-ABC and S'-ABC be two tetrahedrons symmetrical with respect to the plane of their common face ABC; then are they equivalent.



PROOF. These tetrahedrons have a common base ABC, and equal altitudes HS and HS'; hence they are equivalent.

Ex. 660. Two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.

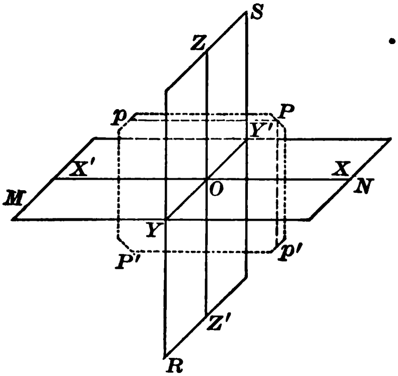
Let A and B be any two symmetrical polyhedrons; we are to prove that they may be decomposed into the same number of tetrahedrons symmetrical each to each.

PROOF. Divide A into tetrahedrons by drawing diagonals from any vertex, and divide B into tetrahedrons by drawing in it the homologous diagonals. Any two corresponding tetrahedrons thus formed will have their vertices symmetrical each to each, and will therefore be symmetrical each to each.

Ex. 661. Two symmetrical polyhedrons are equivalent.

PROOF. Since two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each, and since any two symmetrical tetrahedrons are equivalent, any two symmetrical polyhedrons are equivalent.

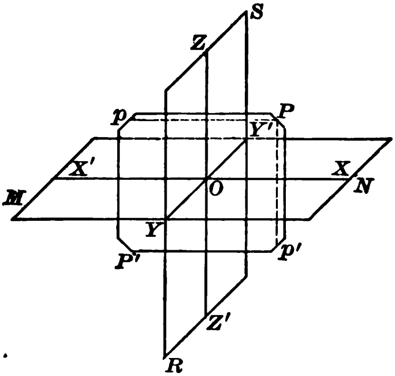
Ex. 662. If a solid has two planes of symmetry perpendicular to each other, the intersection of these planes is an axis of symmetry of the solid.



Let MN and RS be two perpendicular planes of symmetry of any solid; we are to prove that their intersection YY' is an axis of symmetry of the solid.

PROOF. Through any point P of the surface of the solid pass a plane $\perp$ to the intersection YY'; then will the section formed be $\perp$ to each of the planes MN and RS. Hence the points that are the symmetricals of P with respect to each plane lie in this section.

Let p' and p be the symmetricals of P with respect to the planes MN and RS respectively, and let XX' and ZZ' be the intersections of the plane of the section with the planes MN and RS. Then the lines Pp' and Pp are bisected at rt. ∠s by XX' and ZZ' respectively. Hence the section is symmetrical with respect to the rectangular axes XX' and ZZ'. Whence the section is symmetrical with respect to the point O as a centre of symmetry (§ 209).



Now the lines joining symmetrical points of this section with respect to O are bisected at rt. ∠s by YY'. Hence, as P is any point in the surface of the given solid, YY', or the intersection of the planes of symmetry, is an axis of symmetry of the solid.

Ex. 663. If a solid has three planes of symmetry perpendicular to each other, the three intersections of these planes are three axes of symmetry of the solid; and the common intersection of these axes is the centre of symmetry of the solid.

Suppose a solid is symmetrical with respect to the planes A, B, and C, which are ⊥ to each other; we are to prove that their intersections are axes of symmetry, and that their common point is a centre of symmetry.

PROOF. (i.) From Ex. 662, the intersection of the planes A and B is an axis of symmetry. For like reason the intersection of A and C, and that of B and C, are axes of symmetry also.

(ii.) Let P be any point in the surface of the solid. Construct P', the symmetrical of P with respect to the common point of the planes of symmetry, as a centre of symmetry. Through P and P' pass planes parallel to each of the planes of symmetry, thus forming a rectangular parallelopiped. The vertices of this parallelopiped will all be in the surface of the given solid, since any two vertices on the same edge are symmetrical with respect to the plane ⊥ to that edge, and since P is in the surface.

Hence any point P of the surface of the solid has its symmetrical point P' in the surface, and therefore the intersection of the planes of symmetry is a centre of symmetry.

Ex. 664. The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

Let R be the radius, H the altitude, V the volume, and S the lateral area of any right circular cylinder; then by §§ 648, 650,

$$S = 2\pi RH,$$

and
$$V = \pi R^2 H = \tfrac{1}{2} R(2\pi RH) = \tfrac{1}{2} RS.$$

Ex. 665. The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.

Let R be the radius, H the altitude, and V the volume of any right circular cylinder; then R will be the base and H the altitude of the generating rectangle; and since the diagonals of a ▱ intersect midway between its parallel sides, $\tfrac{1}{2} R$ will be the radius of the circumference generated by the point of intersection of the diagonals.

Now by § 363, we have area of rectangle $= RH$.

By § 419, the circumference described by the intersection of the diagonals equals πR.

By § 650, we have $V = \pi R^2 H = (RH)(\pi R)$.

That is, the volume is equal to the product of the area of the generating rectangle by the length of the circumference generated by the point of intersection of its diagonals.

Ex. 666. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to total area multiplied by a third of the radius.

Let R be the radius of the base of a right circular cylinder, $2R$ its altitude, V its volume, and T its total area; we are to prove

$$V = \tfrac{1}{3} RT.$$

Here
$$T = 2\pi R^2 + (2\pi R)\, 2R = 6\pi R^2, \qquad (§ 648)$$

and
$$V = (\pi R^2)\, 2R = \tfrac{1}{3} R(6\pi R^2) = (\tfrac{1}{3} R)\, T. \qquad (§ 650)$$

Ex. 667. Construct a sphere with a given radius whose surface shall pass through three given points.

Let r be any given radius, and A, B, and C any three given points; it is required to construct a sphere with radius r, whose surface shall contain A, B, and C.

ANALYSIS. The centre of the sphere must be at the distance r from each of the points A, B, and C; hence it is in the surfaces of the spheres having A, B, and C as centres, and r as radius.

CONSTRUCTION. Construct the sphere with A as a centre and r as a radius, also the spheres with B and C as centres and r as radius. The spheres having as centres the points common to the surfaces of the three spheres, and the given radius r, will embrace in its surface the three given points.

PROOF. - The distances of the points A, B, and C from either of these common points are equal to the given radius r; hence these points are in the surfaces of spheres having these points as centres and r as radius.

DISCUSSION. If one of the three spheres cuts the intersection of the surfaces of the other two there will be two solutions. If one simply touches the intersection, or the point of contact of the other two, there will be one solution. If the surfaces of the three spheres have no point in common, there will be no solution.

Ex. 668. Construct a sphere with a given radius, tangent to a given plane, and whose surface shall pass through two given points.

Let r be any given radius, MN any given plane, and A and B any two given points; it is required to pass the surface of a sphere with radius r through the points A and B and tangent to the plane MN.

ANALYSIS. The centre of the sphere must be at the distance r from each of the points A and B; hence it is in the surfaces of the spheres having A and B as centres and r as radius. The centre must also be at the distance r from the plane MN; hence it is in one of the planes $\parallel$ to MN at the distance r.

CONSTRUCTION. Construct the sphere with A as a centre and r as a radius, also the sphere with B as a centre and r as a radius. Construct also the planes $\parallel$ to the plane MN at the distance r. The spheres having as centres the points common to the surfaces of the two spheres and either of the two planes, and the given radius r will pass through the two points and be tangent to the given plane.

PROOF. The surface of each of these spheres passes through the points *A* and *B*, since by construction *A* and *B* are at the distance *r* from its centre. The plane *MN* is tangent to each sphere, since by construction *r* is the ⊥ distance of its centre from the plane *MN*.

DISCUSSION. If the intersection of the spheres *A* and *B* lies in one of the planes ∥ to *MN*, the problem is indeterminate. If one of the planes ∥ to *MN* cuts the intersection of the two spherical surfaces there will be two solutions. If one of these planes touches this intersection there will be one solution. If the two spheres have no point in common with either plane there will be no solution. If the points are on opposite sides of the plane, or if both are in the plane, there will be no solution.

Ex. 669. Construct a spherical surface with a given radius, passing through two given points and tangent to a given sphere.

Let *r* be any given radius, *A* and *B* any two given points, and *O* the centre of a sphere whose radius is *R*; it is required to pass the surface of a sphere with radius *r* through the points *A* and *B* and tangent to the sphere *O*.

ANALYSIS. The centre of the sphere must be at the distance *r* from each of the points *A* and *B*; hence it is in the surfaces of the spheres having *A* and *B* as centres and *r* as radius. That the constructed sphere may be tangent to the given sphere the distance between their centres must equal the sum or difference of their radii.

CONSTRUCTION. Construct the spheres with *A* and *B* as centres and *r* as a common radius. Construct also the spheres with *O* as a common centre and with $r + R$ and $r \sim R$ as radii. The spheres having as centres the points common to the surfaces of the first two spheres and that of either of the second two, and the given radius *r*, will pass through the two points and be tangent to the given sphere.

PROOF. The surface of each of these spheres will pass through the points *A* and *B*, since by construction *A* and *B* are at the distance *r* from its centre. The sphere *O* and each of the spheres drawn will be tangent, since by construction the distance between their centres is equal to the sum or the difference of their radii.

DISCUSSION. There will be four, three, two, one, or no solution, according as the spherical surfaces *A* and *B* have four, three, two, one, or no points in common with the constructed spherical surfaces having *O* as their centre.

Ex. 670. Construct a spherical surface with a given radius passing through a given point and tangent to two given planes.

Let r be any given radius, A any given point, and MN and PQ any two given planes; it is required to pass the surface of a sphere with radius r through the point A and tangent to the planes MN and PQ.

ANALYSIS. The centre of the sphere must be in the surface of the sphere having A as a centre and r as a radius. It must also be in the plane ∥ to the plane MN at the distance r, on the side on which A lies, and in the plane ∥ to the plane PQ at the distance r on the A side.

CONSTRUCTION. Construct the sphere with A as centre and r as radius. Construct the plane ∥ to the plane MN at the distance r on the A side; also the plane ∥ to the plane PQ at the distance r on the A side. The spheres having as centres the points common to the intersection of these constructed planes and the surface of the constructed sphere, and the given radius r will pass through the point A and be tangent to each of the planes MN and PQ.

PROOF. The surface of each of these spheres will pass through the point A, since by construction A is at the distance r from its centre. Each of these spheres will be tangent to each of the planes MN and PQ, since by construction r is the perpendicular distance from its centre to each of the planes MN and PQ.

DISCUSSION. (i.) When the two given planes are not parallel: (1) If the given point lies in either of the dihedral ∡ formed by the given planes, there will be two, one, or no solution, according as the surface of the constructed sphere has two, one, or no points in common with the intersection of the constructed planes. (2) If the given point is in either given plane, there will be a solution if the ⊥ to the given plane at the given point cuts either intersection of the constructed planes at the distance r from the given point. Otherwise there will be no solution.

(ii.) When the two given planes are parallel: If the distance between the two planes is $2r$, and the given point lies between them, there will be an unlimited number of spheres fulfilling the given conditions; that is, the problem will be indeterminate. If the given point lies in either of the parallel planes, there will be one, and only one, solution.

There will be no solution when the point does not lie between the planes, or in one of them, or when their distance apart is not $2r$.

Ex. 671. Construct a spherical surface with a given radius passing through a given point and tangent to two given spheres.

Let r be any given radius, A any given point, and O and O' the centres of any two given spheres whose radii are R and R' respectively; it is required to pass the surface of a sphere with radius r through the point A and tangent to the spheres O and O'.

CASE I. *Point without both spheres, neither of which is within the other.*

CONSTRUCTION. Construct the sphere with A as centre and r as radius. Construct also the two spheres with O and O' as centres and $R + r$ and $R' + r$ as radii, respectively. The spheres having as centres the points common to the surfaces of these three constructed spheres, and the given radius r will pass through the point A and be tangent to each of the spheres O and O'.

PROOF. The surface of each of these spheres will pass through the point A, since by construction A is at the distance r from its centre. Each of these spheres will be tangent to each of the spheres O and O', since by construction the distances of its centre from O and O' are $R + r$ and $R' + r$ respectively.

CASE II. *Point and one sphere within the other sphere.*

Suppose the point A and the sphere O are within the sphere O'.

CONSTRUCTION. Construct the sphere with A as centre and r as radius. Construct also the two spheres with O and O' as centres and $R + r$ and $R' - r$ as radii, respectively. The spheres having as centres the points common to the surfaces of these three constructed spheres, and the given radius will pass through the point A and be tangent to each of the spheres O and O'. The proof is similar to that in Case I.

CASE III. *Point in surface of one sphere.*

This will fall under Case I. or II., according as the second sphere lies without or within the first. If the point lies within one sphere, and the other sphere lies without, or *vice versa*, there will be no solution.

Ex. 672. Construct a spherical surface with a given radius passing through a given point and tangent to a given plane and a given sphere.

Let r be any given radius, A any given point, MN any given plane, and O the centre of any given sphere whose radius is R; it is required to pass the surface of a sphere with radius r through the point A and tangent to the plane MN and the sphere O.

CONSTRUCTION. Construct the sphere with A as a centre and r as a radius. Construct also the spheres with O as a centre and $R + r$ and $R - r$ as radii, respectively; and construct the planes $\parallel$ to the given plane and at the distance r from it. The spheres having as centres the points common to the sphere A, either of the constructed parallel planes, and either of the spheres constructed with O as centre, and the given radius r, will pass through the point A, be tangent to the given sphere O and the given plane MN.

PROOF. The surface of each of these spheres will pass through the point A, since by construction A is at the distance r from its centre. Each of these spheres will be tangent to the plane MN, since by construction r is the perpendicular distance from its centre to the plane MN. And each will be tangent to the given sphere O, since by construction the distance of its centre from O is $R + r$ or $R - r$.

Ex. 673. Construct a sphere with a given radius and tangent to three given planes.

Let r be any given radius, and MN, PQ, and RS any three given planes; it is required to construct a sphere with radius r and tangent to the three planes MN, PQ, and RS.

CONSTRUCTION. Construct the two planes ∥ to the plane MN and at the distance r from it. Construct also the planes ∥ to the planes PQ and RS respectively, and at the distance r from each. The spheres having as centres the points common to any three of the six constructed planes, and the given radius r, will be tangent to each of the three given planes.

PROOF. Each of these spheres will be tangent to each of the given planes, since by construction r is the perpendicular distance from its centre to each of the planes MN, PQ, and RS.

DISCUSSION. (i.) If the given planes have but a single point in common, there will be eight different spheres.

(ii.) If the given planes have a line in common, or if they are parallel, there will be no solution.

Ex. 674. Construct a sphere with a given radius and tangent to three given spheres.

Let r be any given radius, and O_1, O_2, and O_3 the centres of any three given spheres, whose radii are R_1, R_2, and R_3 respectively; it is required to construct a sphere with radius r and tangent to the three given spheres O_1, O_2, and O_3.

CONSTRUCTION. Construct the sphere with O_1 as a centre and $R_1 + r$ as a radius; construct also the spheres with O_2, and O_3 as centres and $R_2 + r$ and $R_3 + r$ as radii, respectively. The spheres having as centres the points common to the surfaces of these three constructed spheres and the given radius r will be tangent to each of the three given spheres O_1, O_2, and O_3.

Proof. Each of these spheres will be tangent to each of the given spheres, for the distance between the centre of each and each of the given spheres is equal to the sum of their radii.

Discussion. There will be two, one, or no solution according as the surfaces of the three constructed spheres have two, one, or no points in common. If the three constructed spheres intersect in the same ⊙, the problem is indeterminate. If one of the given spheres lies within, and another without, the third, the problem is impossible. If two of the given spheres lie within the third O_3, construct the spheres with O_1, O_2, and O_3 as centres and R_1+r, R_2+r, and R_3-r as radii, respectively. The points common to the surfaces of these three spheres will be the centres of the required spheres.

Ex. 675. Construct a sphere with a given radius and tangent to two given planes and a given sphere.

Let r be any given radius, MN and PQ any two given planes, and O the centre of a sphere whose radius is R; it is required to construct a sphere with radius r and tangent to each of the planes MN and PQ, and to the sphere O.

Construction. Construct the spheres with O as a common centre, and $R+r$ and $R \sim r$ as radii, respectively. Construct the planes ∥ to the plane MN at the distance r; also construct the planes ∥ to the plane PQ at the distance r. The spheres having as centres the points common to either two of these constructed planes and either one of the constructed spheres, and the given radius r will be tangent to each of the planes MN and PQ and to the sphere O.

Proof. Each of these spheres will be tangent to each of the planes MN and PQ, since by construction r is the perpendicular distance from its centre to each of these planes. Each will be tangent to the sphere O, since the distance between its centre and the centre O is $R+r$ or $R \sim r$.

Discussion. (i.) When the two given planes intersect. If the intersection of the two planes passes through the given sphere there *may* be as many as sixteen different spheres constructed. If both planes do not cut the sphere there may be two, one, or no solution.

(ii.) When the two given planes are parallel. If the distance between the planes is $2r$, the problem is indeterminate, has one or no solution, according as the plane midway between the given planes cuts, is tangent to, or has no point in common with the surface of the sphere whose radius is $R+r$. If the distance between the planes is greater or less than $2r$ there will be no solution.

Ex. 676. Construct a sphere with a given radius and tangent to two given spheres and a given plane.

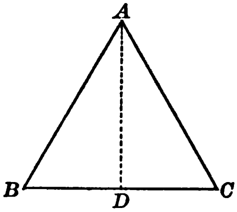
Let r be any given radius, MN any given plane, and O_1 and O_2 the centres of two spheres whose radii are R_1 and R_2 respectively; it is required to construct a sphere with radius r and tangent to the plane MN and the spheres O_1 and O_2.

.Construction. Construct the planes ∥ to the plane MN at the distance r; also construct the spheres with O_1 and O_2 as centres and $R_1 + r$ and $R_2 + r$ as radii, respectively. The spheres having as centres the points common to either of the two planes and both the constructed spheres, and the given radius r, will be tangent to the plane MN and the sphere O_1 and O_2.

Proof. Each of these spheres will be tangent to the plane MN, since by construction r is the perpendicular distance of its centre from this plane. Each will be tangent to the spheres O_1 and O_2, since the distances of its centre from O_1 and O_2 are $R_1 + r$ and $R_2 + r$ respectively.

Discussion. If the given spheres are not on opposite sides of the given plane there may be two, one, or no solutions. If the given spheres are on opposite sides of the given plane the problem is impossible, except when they are tangent to the given plane at the same point, when there are two solutions. If one sphere is within the other the problem is impossible, except when the given plane is tangent to or cuts the greater sphere. If the intersection of the constructed spheres O_1 and O_2 coincides with either plane ∥ to MN, the problem is indeterminate.

Ex. 677. Find the area of a solid generated by an equilateral triangle turning about one of its sides, if the length of the side is a.



Let ABC be an equilateral △ whose sides are each a. It is required to find the area of the solid generated by the revolution of this △ about BC as an axis.

Solution. Let fall the ⊥ from A to the side BC; then, since the △ ABC is equilateral,

$$DB = \tfrac{1}{2}a, \qquad \text{(Ex. 16)}$$

and

$$AD = \sqrt{\overline{AB}^2 - \overline{BD}^2} = \tfrac{1}{2}a\sqrt{3}. \qquad (§\ 339)$$

Now it is evident that the area of the solid generated by the revolution of the △ ABC about BC is twice the convex surface of the cone generated by the revolution of the rt. △ ABD about BD.

Let S denote the convex surface of any cone of revolution, R the radius of its base, and L its slant height; then by § 671 we have

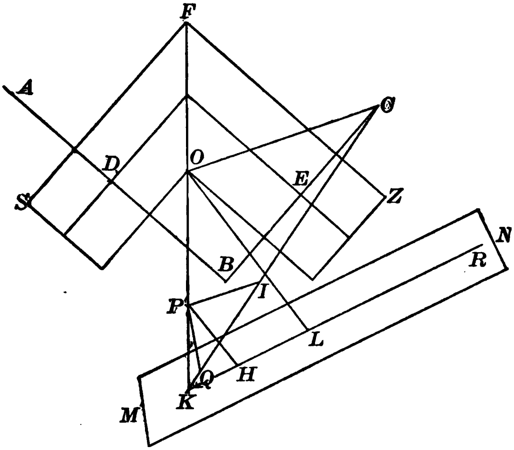
$$S = \pi RL. \qquad (1)$$

Here $L = a$, and $R = \frac{1}{2}a\sqrt{3}$. Substituting these values in (1), we obtain

$$S = \tfrac{1}{2}\pi a^2\sqrt{3}.$$

Hence the area of the surface of the solid generated by the revolution of the △ ABC about BC as an axis equals $\pi a^2\sqrt{3}$.

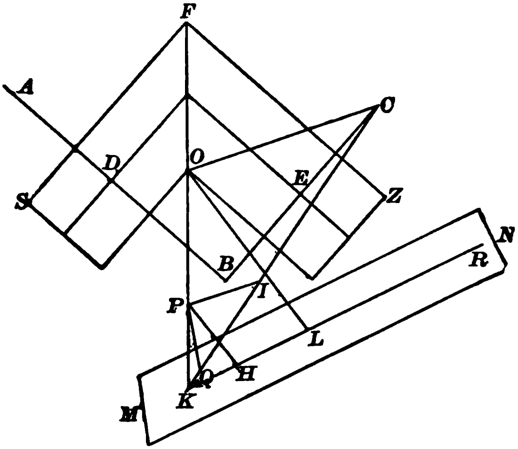
Ex. 678. Find the centre of a sphere whose surface shall pass through three given points and shall touch a given plane.



Let A, B, and C be any three given points in space, and MN any given plane. It is required to find the centre of a sphere whose surface shall pass through A, B, and C, and be tangent to the plane MN.

Analysis. The centre must be equally distant from A and B; hence it must be in the plane ⊥ to the line AB at its middle point. For like reason it must be in the plane ⊥ to BC at its middle point. Therefore the centre is in the line of intersection of these two planes.

Again, the centre must be equally distant from the point C and the plane MN; hence it is the point in this intersection that is equally distant from the point C and the plane MN.



Construction. Draw AB and BC, and bisect each of them. Through the middle points D and E pass the planes SOF and ZOF ⊥ to AB and BC respectively. Let FP be the intersection of these planes, which meets MN in K. Through FP pass a plane ⊥ to the plane MN, intersecting it in KR. Draw CK, and from any point P in KF draw PH ⊥ to KR. With P as a centre and a radius equal to PH, strike an arc in plane FKC, cutting KC in the points I and Q. Join P and I. Through C draw $CO \parallel$ to IP, and let O be its intersection with FP; then O is the centre of the spherical surface through A, B, and C, and tangent to the plane MN, and OC is its radius.

Proof. The point O is equally distant from the points A, B, and C (any point in a plane ⊥ to a line at its middle point is equally distant from its extremities). From O draw OL ⊥ to KR; then will it be ⊥ to the plane MN (§ 515). The lines PH and OL will be parallel (§ 100). Hence the △ KPH and KOL are similar, and we have

$$KP : KO = PH : OL. \qquad (1)$$

The △ KPI and KOC are also similar; hence

$$KP : KO = PI : OC. \qquad (2)$$

Since $PH = PI$, from (1) and (2) we have

$$OL = OC.$$

Hence the spherical surface with the centre O, and with the radius OL or OC, will pass through the three points A, B, and C, and be tangent to the plane MN.

Discussion. There is no solution if two points are in the plane or on opposite sides of it. When one point, as A, is in the plane, there is one or no solution, according as the line ⊥ to MN at A cuts or does not cut the intersection of the planes ⊥ to AB and BC at their middle points.

When the three points are on the same side of the plane there will be two, one, or no solution, according as the $\angle CKF$ is less, equal to, or greater than the $\angle FKR$.

Ex. 679. Find the centre of a sphere whose surface shall touch two given planes and also pass through two given points which lie between the planes.

Let A and B be any two points between the planes MN and PQ; it is required to find the centre of a spherical surface which shall pass through the points A and B and be tangent to the planes MN and PQ.

Analysis. The centre must be equally distant from A and B, and therefore in the plane ⊥ to AB at its middle point. The centre must be equally distant from the planes MN and PQ, and therefore in the plane bisecting the dihedral ∠ included between them. Hence the centre is in the line of intersection of these two planes.

Again, the centre must be equally distant from the point B and the plane MN; hence it is the point in this line of intersection that is equally distant from the point B and the plane MN.

Construction. Bisect the ∠ between the planes, draw AB, and bisect it with a perpendicular plane. Let FK be the intersection of this plane with the plane bisecting the dihedral ∠ between the given planes. Find, as in Ex. 678, the point O in FK at equal distances from B and the plane MN; then is O the centre of the spherical surface through the points A and B, and tangent to the planes MN and PQ.

Proof. Since O is in the plane bisecting the ∠ included between the given planes, it is equally distant from these planes (§ 525).

Since O is in the plane ⊥ to AB at its middle point, it is equally distant from A and B (§ 482).

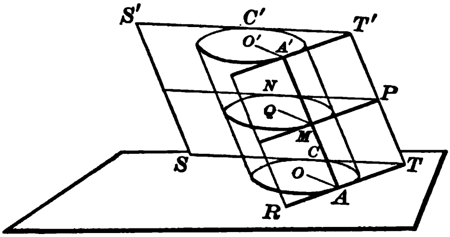
By construction O is equally distant from the point B and the plane MN.

Hence O is equally distant from the points A and B and the planes MN and PQ, and is therefore the centre of the spherical surface which passes through A and B and is tangent to the planes MN and PQ.

Discussion. If the given planes are parallel, instead of the bisecting plane, we draw a plane midway between the parallel planes. As in Ex. 678, when the points are all on the same side of the plane, there may be two, one, or no solution.

Ex. 680. Through a given point to pass a plane tangent to a given circular cylinder.

Case I. *When the given point is in the curved surface of the cylinder.*



Let AC' be a given circular cylinder, and let the given point be a point in the element AA'; it is required to pass a plane tangent to the cylinder and embracing the element AA'.

Construction. Draw the radius OA, and AT tangent to the base; and pass a plane RT' through AA' and AT. The plane RT' is the plane required.

Proof. Through any point P in this plane, not in the element AA', pass a plane ∥ to the base, intersecting the cylinder in the ⊙ MN, and the plane RT' in MP. From the centre of the ⊙ MN draw QM. MP and MQ are ∥ respectively to AT and AO (the intersections of two ∥ planes by a third plane are ∥ lines);

$$\therefore \angle PMQ = \angle TAO, \qquad (\S\ 498)$$

(two ∠s not in the same plane, having their sides ∥ and lying in the same direction, are equal). Therefore PM is tangent to the ⊙ MN at M (§ 239). Therefore P lies without the ⊙ MN, and hence without the cylinder. Therefore the plane RT' contains the element AA' but does not cut the cylinder, and is therefore tangent to it.

Case II. *When the given point is without the cylinder.*

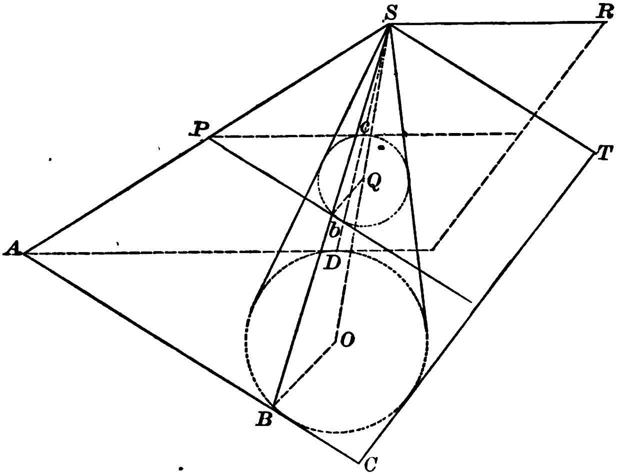
Let P be the given point; it is required to pass a plane through P tangent to the cylinder.

Construction. Through P draw the line PT ∥ to the elements of the cylinder, meeting the plane of the base at T. From T draw TA and TC tangents to the base (§ 287). The planes ATP and CTP are the required tangent planes.

Proof. Since AA' is ∥ to PT (cons.), the plane RT', passing through PT and the point A, will contain the element AA' (two ∥ lines lie in the same plane). And, since RT' also contains the tangent AT, it is a tangent plane to the cylinder (Case I.). In like manner the plane TS' is shown to be a tangent plane to the cylinder.

Ex. 681. Through a given point to pass a plane tangent to a given circular cone.

Case I. *When the given point is in the curved surface of the cone.*



Let S-BD be a given circular cone, and let the given point be a point in the element BS; it is required to pass a plane tangent to the cone and embracing the element SB.

Construction. Draw the radius OB, and the tangent AB to the base at B, and pass a plane through SB and AB; then is the plane ABS the plane required.

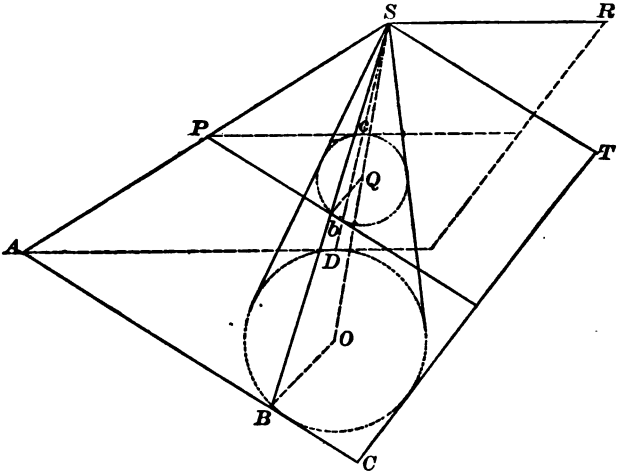
Proof. Through any point P in this plane, not in the element BS, pass a plane ‖ to the base, intersecting the cone in the ⊙ bc, and the plane ABS in Pb (§§ 668, 471).

From the centre Q of the ⊙ bc draw Qb. Pb and Qb are ‖ respectively to AB and OB (the intersections of two ‖ planes by a third plane are ‖ lines).

$$\therefore \angle PbQ = \angle ABO. \qquad (§\ 498)$$

Hence Pb is tangent to the ⊙ bc at b (§ 239). Whence P lies without the ⊙ bc, and therefore without the cone.

Hence the plane ABS contains the element BS but does not cut the cone, and is therefore a tangent plane to it.



Case II. *When the given point is without the cone.*

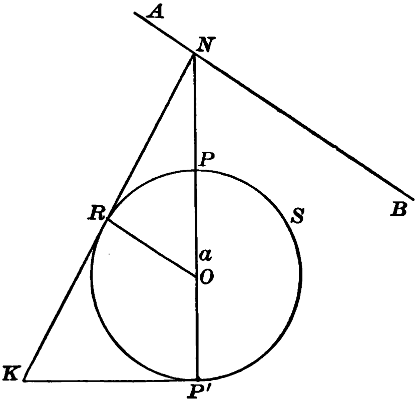
Let P be the given point without the cone S-BD; it is required to pass a plane through P tangent to the cone.

Construction. Join the vertex S and the point P, and produce SP to meet the plane of the base in some point, as A. From A draw AB and AD tangents to the base, and through SA and these tangents pass the planes ABS and ADS; then are ABS and ADS the planes required.

Proof. Since the lines SB and SD are elements of the cone, each of the planes ABS and ADS is the plane of an element and a tangent to the base at the foot of the element. Hence, by the proof in Case I., each of these planes is tangent to the cone.

Cor. When the given point is without the cone the problem may be stated as follows: Through a given line passing through the vertex of a cone to pass a plane tangent to the cone.

Ex. 682. Through a given straight line without a given sphere to pass a plane tangent to the sphere.



Let AB be any given straight line without the sphere whose centre is O; it is required to pass a plane through AB tangent to the sphere whose centre is O and radius OR.

Construction. Through the line AB and the centre O pass a plane. Let this plane be represented by the plane of the paper. Its intersection with the surface of the sphere will be a great ⊙ of the sphere.

Let $P'SPR$ be this great ⊙. From any point N in AB draw NRK, a tangent to the ⊙ $P'SPR$. Through N and O draw NOP', and through P' draw $P'K \perp$ to $P'N$.

While AB remains fixed, conceive the figure NKP' to revolve about NP' as an axis. The semicircle PRP' will generate the given sphere, and the $\triangle$ NKP' will generate the circumscribed circular cone. By Ex. 681, Cor., through the line AB pass planes tangent to this cone; then will these two planes be the required tangent planes to the sphere.

Proof. Every element of the cone has one, and only one, point in common with the surface of the sphere. Hence any plane tangent to the cone and embracing AB and only one of these elements has one, and only one, point in common with the surface of the sphere, and is therefore, by definition, a tangent plane to the sphere.

Ex. 683. The volume of a sphere is two-thirds of the volume of a circumscribing cylinder, and its surface is two-thirds of the total surface of the cylinder.

Let V denote the volume, and S the surface, of any sphere; also let V' denote the volume, and T the total surface, of the circumscribing cylinder. It is required to prove that

$$V = \tfrac{2}{3}V', \text{ and } S = \tfrac{2}{3}T.$$

By §§ 785, 650, we have, since $H = 2R$,

$$V = \tfrac{4}{3}\pi R^3, \text{ and } V' = 2\pi R^3.$$

$$\therefore \frac{V}{V'} = \frac{\tfrac{4}{3}\pi R^3}{2\pi R^3} = \frac{2}{3},$$

or $$V = \tfrac{2}{3}V'.$$

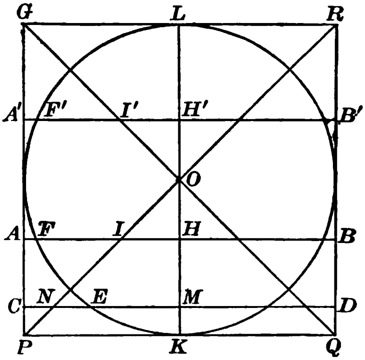
By §§ 764, 648 we have

$$S = 4\pi R^2, \text{ and } T = 6\pi R^2.$$

$$\therefore \frac{S}{T} = \frac{4\pi R^2}{6\pi R^2} = \frac{2}{3},$$

or $$S = \tfrac{2}{3}T.$$

Ex. 684. Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder; if any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindrical and conic segments.



Let LFK be any ⊙, $RGPQ$ a circumscribed square, RP and GQ its diagonals, LK a diameter of the ⊙ ∥ to RQ, and AB and CD any two lines cutting the ⊙ ⊥ to LK. Then, if the figure be revolved about LK as an axis, the ⊙ will generate a sphere, the square the circumscribed cylinder, the $\triangle$ ORG and OPQ the inscribed cone of two nappes, and AB and CD the planes ⊥ to the axis LK. We are to prove that, when the figure is revolved about LK, the spherical segment generated by $HFEM$ is equal to the difference between the cylinder generated by $HACM$ and the frustum of a cone generated by $HINM$.

PROOF. Let $R = OK,\ a = OH,\ b = OM.$

The spherical segment generated by $HMEF$ is equal to the difference between the spherical segments generated by $HKEF$ and MKE.

If h denote the altitude of a spherical segment of one base, from § 790 we have for the volume of a spherical segment of one base

$$V = \pi h^2 (R - \tfrac{1}{3} h).$$

If $h = HK = R - a$,

$$\begin{aligned} V &= \pi (R-a)^2 [R - \tfrac{1}{3}(R-a)] \\ &= \pi (\tfrac{2}{3} R^3 - aR^2 + \tfrac{1}{3} a^3). \end{aligned} \tag{1}$$

If $h = MK = R - b$,

$$\begin{aligned} V &= \pi (R-b)^2 [R - \tfrac{1}{3}(R-b)] \\ &= \pi (\tfrac{2}{3} R^3 - bR^2 + \tfrac{1}{3} b^3). \end{aligned} \tag{2}$$

Subtracting (2) from (1), we obtain

$$\text{Vol } HFEM = \pi R^2 (b-a) - \tfrac{1}{3}\pi (b^3 - a^3). \tag{3}$$

Since $MC = R$, and $MH = OM - OH = b - a$, we have, by § 650,

$$\text{Vol } HMCA = \pi R^2 (b-a). \tag{4}$$

Since $PK = OK$, $NM = MO = b$, and $IH = OH = a$. The volume generated by $HMNI$ is equal to the cone generated by OMN diminished by the cone generated by OHI.

$$\text{Cone } OMN = \tfrac{1}{3} MO \times \pi \overline{NM}^2 = \tfrac{1}{3} \pi b^3. \qquad (\text{§ } 673)$$

$$\text{Cone } OHI = \tfrac{1}{3} OH \times \pi \overline{IH}^2 = \tfrac{1}{3} \pi a^3.$$

$$\therefore \text{Vol } HMNI = \tfrac{1}{3} \pi (b^3 - a^3). \tag{5}$$

From (3), (4), and (5), we obtain

$$\text{Vol } HFEM = \text{vol } HMCA - \text{vol } HMNI.$$

When the planes are on opposite sides of the centre O, as those generated by $A'B'$ and CD, the volume of the spherical segment generated by $H'F'EM$ is equal to the volume of the entire sphere diminished by the volume of the segments generated by EMK and $F'H'L$ respectively; hence, if $OH' = a$, we have

$$\begin{aligned} \text{Vol } H'F'EM &= \tfrac{4}{3}\pi R^3 - \pi (\tfrac{2}{3} R^3 - aR^2 + \tfrac{1}{3} a^3) - \pi (\tfrac{2}{3} R^3 - bR^2 + \tfrac{1}{3} b^3) \\ &= \pi R^2 (a+b) - \tfrac{1}{3}\pi (a^3 + b^3) \\ &= \text{vol } MH'A'C - \text{vol } OH'I' - \text{vol } OMN. \end{aligned}$$

Ex. 685. Compare the volumes of the solids generated by a rectangle turning successively about two adjacent sides, the lengths of these sides being a and b.

The volume generated in either case will be a cylinder of revolution. Let V denote the volume, H the altitude, and R the radius of the base of any cylinder; then, by § 650, we have

$$V = \pi R^2 \times H. \qquad (1)$$

If the rectangle be revolved about the side a, $H = a$, and $R = b$; hence, denoting the volume generated by V_1, from (1) we have

$$V_1 = \pi b^2 a. \qquad (2)$$

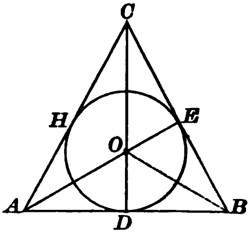
If the rectangle be revolved about the side b, $H = b$, and $R = a$; hence, denoting the volume generated by V_2, from (1) we have

$$V_2 = \pi a^2 b. \qquad (3)$$

Dividing (2) by (3), we obtain

$$\frac{V_1}{V_2} = \frac{\pi b^2 a}{\pi a^2 b} = \frac{b}{a}.$$

Ex. 686. An equilateral triangle revolves about one of its altitudes. Compare the convex surface of the cone generated by the triangle and the surface of the sphere generated by the circle inscribed in the triangle.



Let HDE be the ⊙ inscribed in the equilateral △ ABC, each of whose sides is a; it is required to find the ratio of the surface generated by the sides AC and BC to that generated by the circumference of the ⊙ when the figure is revolved about the altitude CD.

Solution. Since the △ is equilateral, each of the altitudes is both a medial line of the △ and a bisector of an ∠; hence each of the altitudes CD and AE bisects the △ and passes through the centre of the ⊙. The lines OD and OE are radii drawn to the points of contact of the tangents AB and BC (they are lines from the centre ⊥ to tangents).

$$OD = \tfrac{1}{3} CD$$

(medial lines of a △ trisect each other).

But $\qquad CD = \sqrt{\overline{CB}^2 - \overline{DB}^2} = \tfrac{1}{2} a\sqrt{3}.$

Hence $\qquad OD = \tfrac{1}{6} a\sqrt{3}. \qquad (1)$

By § 671, if $S =$ convex surface, $R =$ radius of base, and $L =$ slant height of a cone of revolution, we have

$$S = \pi RL.$$

Here $R = \frac{1}{2}a$, and $L = a$; hence

$$S = \tfrac{1}{2}\pi a^2. \qquad (2)$$

By § 764, if $S_1 =$ surface, and $R =$ radius, of a sphere,

$$S_1 = 4\pi R^2.$$

From (1), $R = \frac{1}{6}a\sqrt{3}$;

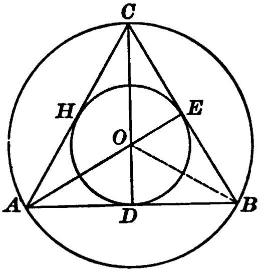
hence $S_1 = 4\pi(\frac{1}{6}a\sqrt{3})^2 = \frac{1}{3}\pi a^2. \qquad (3)$

Dividing (2) by (3), we obtain

$$\frac{S}{S_1} = \frac{\frac{1}{2}\pi a^2}{\frac{1}{3}\pi a^2} = \frac{3}{2};$$

that is, the surfaces are in the ratio 3 : 2.

Ex. 687. An equilateral triangle revolves about one of its altitudes. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.



Let *ABC* be an equilateral △, each of whose sides is a; and let *HDE* be the inscribed, and *ABC* the circumscribed ⊙. It is required to find the ratio of the solids generated by the △, the inscribed ⊙, and the circumscribed ⊙, when the figure is revolved about the altitude *CD*.

We have proved in Ex. 686 that the altitude *CD* bisects the equilateral △ and passes through the centre of both the Ⓢ, and that the radius of the inscribed ⊙ is $\frac{1}{6}a\sqrt{3}$.

By § 673, if V denote the volume, R the radius of the base, and H the altitude of a cone of revolution,

$$V = \tfrac{1}{3}\pi R^2 H.$$

Here $R = \frac{1}{2}a$, and $H = \frac{1}{2}a\sqrt{3}$;

hence $V = \frac{1}{24}\pi a^3\sqrt{3}. \qquad (1)$

By § 785, if V denote the volume, and R the radius, of a sphere,

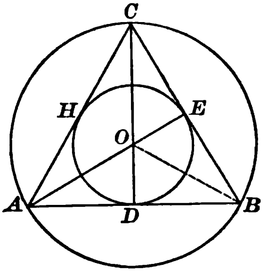
$$V = \tfrac{4}{3}\pi R^3.$$

Let V_1 denote the volume generated by the inscribed ⊙; then

$$R = \tfrac{1}{6}a\sqrt{3},$$

and $V_1 = \frac{4}{3}\pi(\frac{1}{6}a\sqrt{3})^3 = \frac{\pi}{54}a^3\sqrt{3}. \qquad (2)$

Let V_2 denote the volume generated by the circumscribed ⊙; then



$$R = OB = \sqrt{\overline{OD}^2 + \overline{DB}^2}$$
$$= \sqrt{\tfrac{1}{12}a^2 + \tfrac{1}{4}a^2} = \tfrac{1}{3}a\sqrt{3},$$

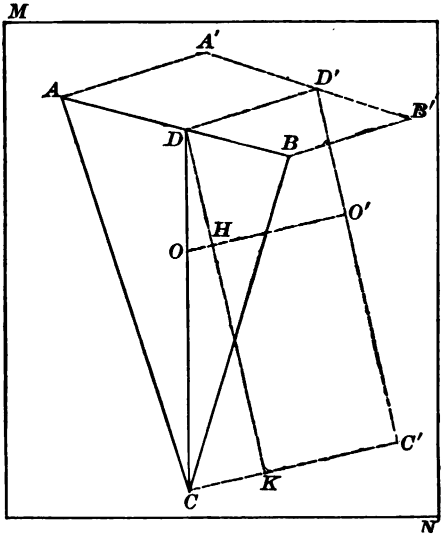
and $$V_2 = \tfrac{4}{3}\pi\left(\tfrac{1}{3}a\sqrt{3}\right)^3 = \frac{4\pi a^3}{27}\sqrt{3}. \quad (3)$$

From (1), (2), and (3), we obtain

$$V : V_1 : V_2 = \tfrac{1}{24}\pi a^3\sqrt{3} : \tfrac{1}{54}\pi a^3\sqrt{3} : \tfrac{4}{27}\pi a^3\sqrt{3}$$
$$= 9 : 4 : 32.$$

That is, the volumes are proportional to 9, 4, and 32.

Ex. 688. The perpendicular let fall from the point of intersection of the medial lines of a given triangle upon any plane not cutting the triangle is equal to one-third the sum of the perpendiculars from the vertices of the triangle upon the same plane.



Let OO' be the ⊥ from O, the intersection of the medial lines of the △ ABC, to the plane MN which does not cut the △, and let AA', BB', and CC' be the ⊥s from the vertices of the △ to the same plane; we are to prove that $$OO' = \tfrac{1}{3}(AA' + BB' + CC').$$

Proof. Suppose CC' to be the greatest of these $\perp$s. Draw $DD' \perp$ to the plane from the middle point D of the side AB, and through D draw, in plane DCC', $DK \parallel$ to the plane MN.

Now AA', DD', and BB' will all lie in the same plane, whose intersection with MN is $A'D'B'$. In the trapezoid $AA'B'B$,

$$AA' + BB' = 2\,DD'$$

(DD' joins the middle points of its non-$\parallel$ sides).

The $\perp$s DD', OO', CC' all lie in the same plane, and

$$2\,DD' + KC' = 3\,HO'$$

($\parallel$s included between $\parallel$s are equal).

$$KC = 3\,OH$$

($\triangle$s DOH and DCK are similar, and $DC = 3\,DO$).

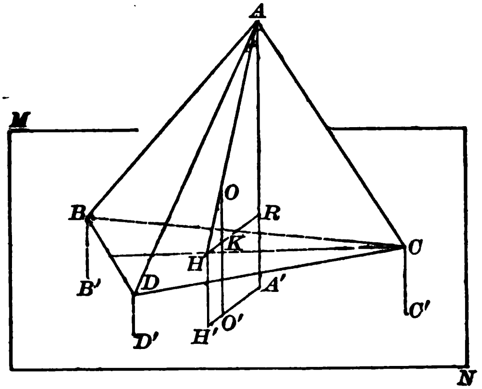
Hence

$$AA' + BB' + CC' = 2\,DD' + KC' + CK = 3(HO' + OH),$$

or

$$OO' = \tfrac{1}{3}(AA' + BB' + CC').$$

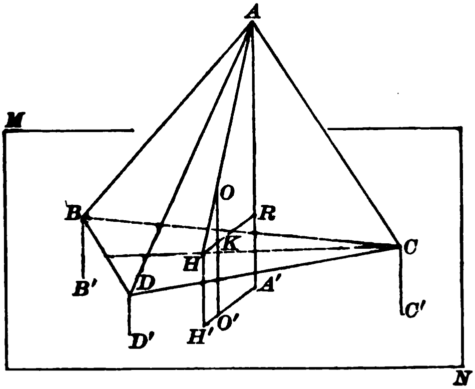
Ex. 689. The perpendicular from the centre of gravity of a tetrahedron upon any plane not cutting the tetrahedron is equal to one-fourth the sum of the perpendiculars from the vertices of the tetrahedron upon the same plane.



Let OO' be the $\perp$ from O, the centre of gravity of the tetrahedron $ABCD$, to the plane MN, which does not cut the tetrahedron, and let AA', BB', CC', and DD' be the $\perp$s from the vertices of the tetrahedron to the same plane. We are to prove that

$$OO' = \tfrac{1}{4}(AA' + BB' + CC' + DD').$$

Proof. Let H be the intersection of the medial lines of the face BDC; then we know (Ex. 648) that the centre of gravity O lies on the line HA at a distance from H equal to $\frac{1}{4} HA$. Through H, in plane HAA', draw $HR \parallel$ to the plane MN cutting OO' at K.



Now $$BB' + CC' + DD' = 3\,HH'. \qquad \text{(Ex. 688)}$$

$$3\,HH' + RA' = 4\,KO'$$

($\parallel$s included between $\parallel$s are equal).

$$AR = 4\,OK$$

($\triangle$ HKO and HRA are similar, and $HA = 4\,HO$).

Hence $$BB' + CC' + DD' + AA' = 3\,HH' + RA' + AR$$
$$= 4(KO' + OK) = 4\,OO',$$

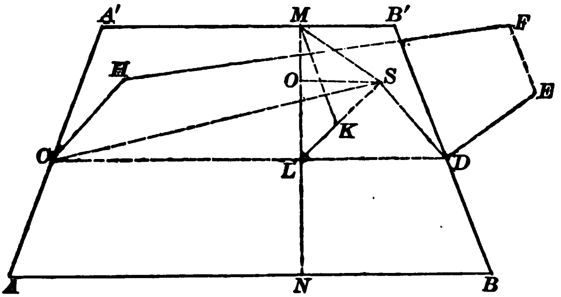
or $$OO' = \tfrac{1}{4}(AA' + BB' + CC' + DD').$$

Ex. 690. The volume of any polyhedron having for its bases any two polygons whose planes are parallel, and for lateral faces trapezoids, is the product of one-sixth the distance between the bases into the sum of the two bases plus four times a section midway between the bases.

Let $ABB'A'$ be a lateral face, and $CDEFH$ a section midway between the bases of such a polyhedron; denote its bases by B and b, the distance between them by H, and the section midway by B'. We are to prove that $$V = \tfrac{1}{6} H(B + b + 4\,B').$$

Proof. Join S, any point in the section, with the vertices of the polyhedron, thus dividing the solid into as many pyramids as it has faces. The volumes of the two pyramids having B and b' as bases are evidently equal to $\frac{1}{6} H \times B$ and $\frac{1}{6} H \times b$, respectively.

We next find the volume of the pyramid having the lateral face $ABB'A'$ as base and S as vertex. From S let fall $SO \perp$ to this face, through O draw $MON \perp$ to AB, or its ∥ CD, cutting CD in L, draw SL, and draw $MK \perp$ to SL.



Since SO is $\perp$ to face AB', the dihedral $S\text{-}OL\text{-}C$ is a right dihedral (§ 518).

Hence, as CL is $\perp$ to MN, CL is $\perp$ to the plane MLS (§ 515).

Hence $C\text{-}LS\text{-}M$ is a right dihedral (§ 518).

Whence MK is $\perp$ to $CDEFH$ (§ 515), and is equal to $\frac{1}{2}H$.

$$\text{Now} \quad \text{Vol}\, S\text{-}ABB'A' = ABB'A' \times \tfrac{1}{3}OS = 2\,CD \times LM \times \tfrac{1}{3}SO \ (\S\ 604)$$
$$= \tfrac{2}{3}CD \times LM \times SO. \qquad (1)$$

The ∠s LMK and OSL are equal (acute ∠s having their sides $\perp$ respectively are equal).

Hence the rt. △s LMK and OSL are similar (mutually equiangular △s are similar).

$$\therefore\ LM : LS = MK : OS,$$

or
$$LM \times OS = LS \times MK.$$

Substituting this value of $LM \times SO$ in (1), we have

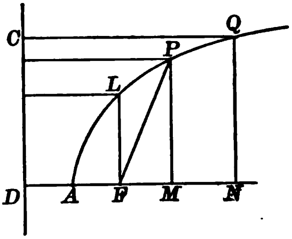
$$\begin{aligned}\text{Vol}\, S\text{-}ABB'A' &= \tfrac{2}{3}CD \times LS \times MK \\ &= (\tfrac{1}{3}MK)\,4(CD \times \tfrac{1}{2}LS) \\ &= (\tfrac{1}{6}H)\,4(CD \times \tfrac{1}{2}LS).\end{aligned}$$

Now $CD \times \frac{1}{2}LS$ is equal to the area of the $\triangle CSD$; hence the volume of a pyramid having a lateral face as its base is equal to $\frac{1}{6}H$ into 4 times the portion of the section B' that is included within its surface, and therefore the sum of the volumes of the pyramids having the lateral faces as bases equals $\frac{1}{6}H(4B')$. Hence we have

$$V = \tfrac{1}{6}H(B + b + B').$$

BOOK IX.

Ex. 691. Prove that if the abscissa of a point is equal to its ordinate, each is equal to the latus rectum.



If $AM = MP$,

then $AM = 4\,AF$.

PROOF.

Since $\overline{MP}^2 = 4\,AF \times AM$, (§ 804)

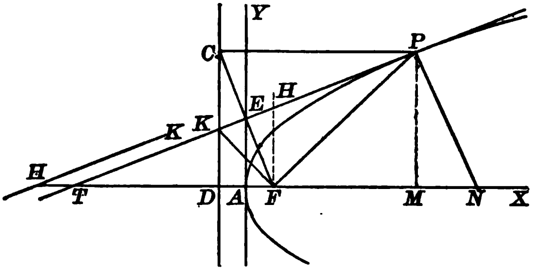
if $MP = AM$;

then $MP = 4\,AF$,

or $AM = MP = 4\,AF$.

Ex. 692. To draw a tangent and a normal at a given point of a parabola.

To draw a tangent and a normal to the parabola AP at any point P.



(i.) Draw the ordinate PM. Lay off $AT = AM$, and join PT; then PT is the tangent required (§ 818).

Draw $PN \perp$ to PT at P; then will PN be the normal (§ 816).

(ii.) Lay off $MN = 2\,AF$, and join PN; then will PN be the normal (§ 819). Draw $PT \perp$ to PN at P; then will PT be the tangent (§ 816).

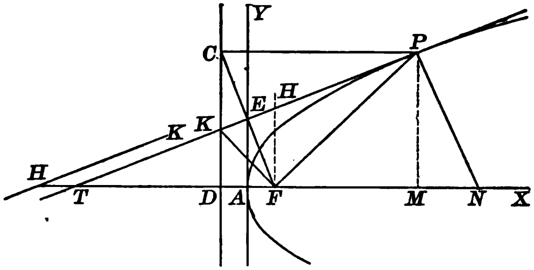
(iii.) Join FP, draw $PC \perp$ to DC, and draw PT bisecting the $\angle\, FPC$; then PT is the tangent at P (§ 811).

Ex. 693. To draw a tangent to a parabola parallel to a given line.

Let HK be any line. To draw a tangent to the parabola AP parallel to the line HK.

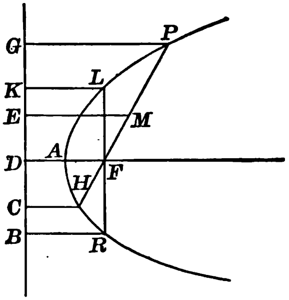
CONSTRUCTION. At F construct the $\angle NFP = 2\angle DHK$. Draw the tangent PT at P. Then will PT be parallel to HK.

PROOF. $\angle NFP = \angle FTP + \angle FTP = 2\angle FTP$. (§ 813)
But $\angle NFP = 2\angle DHK$. (Cons.)
Hence, $\angle FTP = \angle DHK$.
Therefore PT is $\parallel$ to HK.



Ex. 694. Show that the tangents at the ends of the latus rectum meet at D.

PROOF. $FP = FT$. (§ 813)
Now if P is at the end of the latus rectum,
$$FP = 2 \times AF.$$
$$\therefore\ FT = 2 \times AF = FD.$$
Hence the tangents at the ends of the latus rectum both pass through D.



Ex. 695. Prove that the latus rectum is the shortest focal chord.

In the parabola RAP to prove that the latus rectum RL is shorter than any other focal chord HFP.

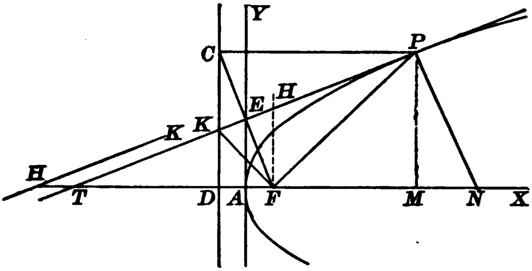
PROOF. Draw PG, LK, HC, and RB, $\perp$ to BG.

Since $GP > CH$, $FP > FH$.

Hence the middle point of HP lies in FP, and therefore to the right of RL.

Let M be this point. Draw $ME \perp$ to BG; then $2\,DF < 2\,EM$. Now $LR = 2\,FD$, and $PH = FP + FH = PG + HC = 2\,EM$. $\therefore LR < PH$.

Ex. 696. The tangent at any point cuts the directrix and the latus rectum produced at points equally distant from the focus.



Let the tangent cut the directrix and latus rectum produced in K and H respectively.

To prove $FK = FH$. Let E be the foot of the $\perp$ from F to the tangent PT; then E lies in AY (§ 815),

Since $AD = AF$, $KE = EH$ (§ 187). $\therefore FK = FH$ (§ 122).

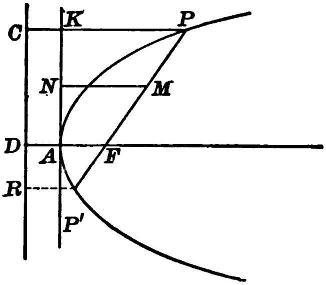
Ex. 697. The circle whose diameter is FP touches the tangent at A.

PROOF. Let M be the middle point of FP. Draw PC, and $MN \perp$ to the tangent at A, which is $\parallel$ to CD (§ 814).

Then $$FP = PC = PK + KC = PK + AF = 2\,NM.$$

$$\therefore FM = NM.$$

Hence the $\odot$ whose centre is M, and radius MF, will be tangent to AK (§ 239).



Ex. 698. The directrix touches the circle having any focal chord for diameter.

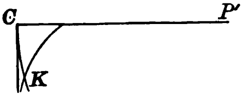
Let PP' be any focal chord. To prove that the $\odot$ whose diameter is $P'P$ is tangent to the directrix DC.

PROOF. $$P'P = FP' + FP = P'R + PC.$$
$$\therefore \tfrac{1}{2} P'P = \tfrac{1}{2}(P'R + PC);$$

that is, $\frac{1}{2} P'P$ equals the distance of the middle point of PP' from the directrix RC (§ 191). Hence the ⊙ on $P'P$ as diameter touches RC (§ 239).

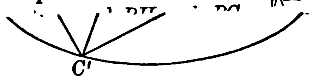
Ex. 699. Given two points and the directrix to find the focus.

Given BC the directrix, and P' and P two points of a parabola, to find its focus.



CONSTRUCTION. From P and P' let fall the ⊥s PH and $P'C$ upon the directrix BC.

[illegible]

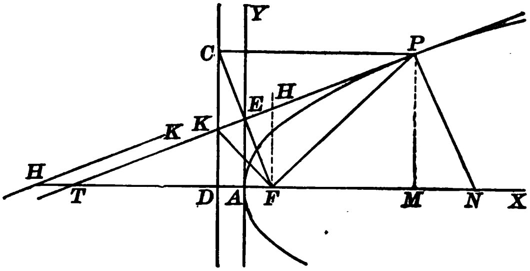


Ex. 717. To draw a diameter conjugate to a given diameter in a giv

Ex. 704. To find the locus of the centre of a circle which passes through a given point and touches a given straight line.

Let F be any given point, and DB any given line. To find the locus of the centre of the ⊙ which shall pass through F and be tangent to DB.

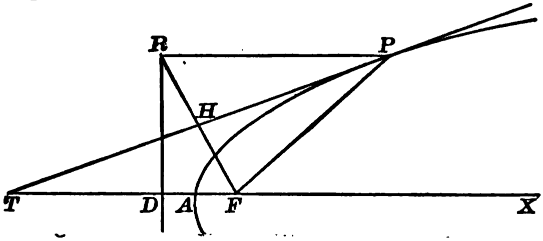
Ex. 700. The ⊥ FE bisects TP.



Since $$FP = FT, \qquad (§ 813)$$
$$EP = ET. \qquad (§ 121)$$

Ex. 701. Given the focus and the axis to describe a parabola which shall touch a given straight line.

Having given the focus F, the axis TX, and a tangent TP; to construct the parabola.



H respectively.

To prove $FK = FH$. Let E be the foot of the ⊥ from F to the tangent PT; then E lies in AY (§ 815),

Since $AD = AF$, $KE = EH$ (§ 187). $\therefore FK = FH$ (§ 122).

Ex. 697. The circle whose diameter is FP touches the tangent at A.

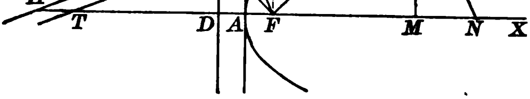
PROOF. Let M be the middle point of FP. Draw PC, and MN ⊥ to the tangent at A, which is ∥ to CD (§ 814).

$$\text{Then} \qquad FP = PC = PK + KC = PK + AF = 2\,NM.$$

$$\therefore FM = NM.$$

Hence the ⊙ whose centre is M and radius MF will be tangent to AK

Ex. 702. If PN is a normal, and triangle PNF is equilateral, then $PF =$ the latus rectum.



PROOF. If △ PNF is equilateral, the ⊥ PM bisects the base; hence

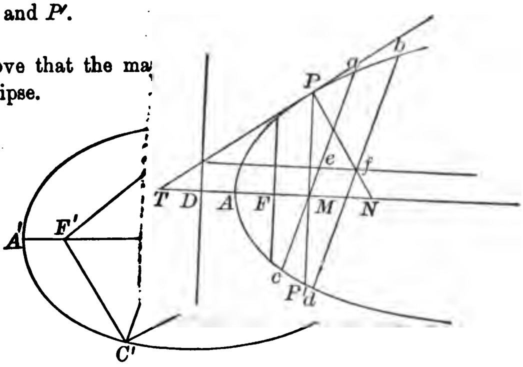
$$FM = MN = \text{semi-latus rectum}.$$

$$\text{Hence } PF = FN = 2\,MN = \text{latus rectum}.$$

Ex. 703. Given a parabola to find the directrix, axis, and focus.

[illegible]
through both P and P'.

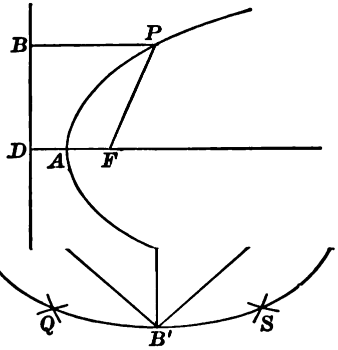
Ex. 707. Prove that the ma
drawn in an ellipse.



Ex. 717. To draw a diameter conjugate to a given diameter in a
giv [illegible]

Ex. 704. To find the locus of the centre of a circle which passes through a given point and touches a given straight line.

Let F be any given point, and DB any given line. To find the locus of the centre of the ⊙ which shall pass through F and be tangent to DB.



PROOF. If $F'BF = 90°$,

$$OM = a,$$

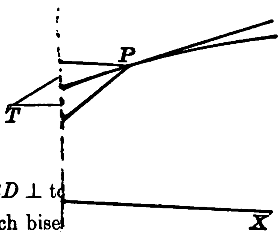
$$ON = e^2 d. \qquad (§ 874)$$

$$\therefore OM : ON = 1 : e^2 = a^2 : a^2e^2 = a^2 : c^2.$$

Ex. 705. Given the axis, a tangent, and the point of contact, to construct the focus and directrix.

Given the axis TX, the tangent TP, and the point of contact P. To construct the focus and the directrix.

X, and a tangent TP; to con-



CONSTRUCTION. Bisect TP, at E the middle point erect REF ⊥ to TP, intersecting TX in F; then is F the focus.

Lay off $ER = EF$, and draw RD ⊥ t

PROOF. By Ex. 700, the ⊥ which bise

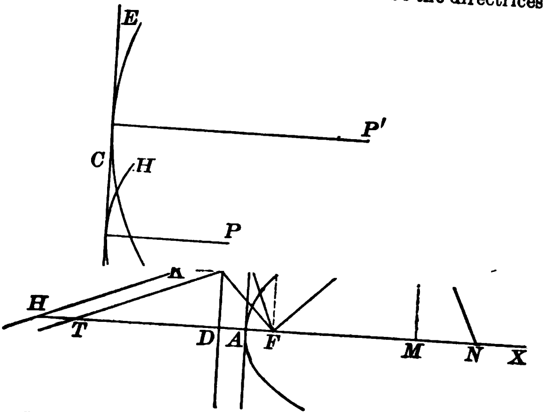
hence F is the focus.

To prove $FK = FH$. Let E be the foot of the ⊥ from F to the tangent PT; then E lies in AY (§ 815),

Since $AD = AF$, $KE = EH$ (§ 187). ∴ $FK = FH$ (§ 122).

find the directrix.

CONSTRUCTION. With P as centre, and PF as radius, draw arc KH. With P' as centre, and $P'F$ as radius, draw arc DE. Draw the common tangents BC and $B'C'$; then BC and $B'C'$ will be the directrices required.

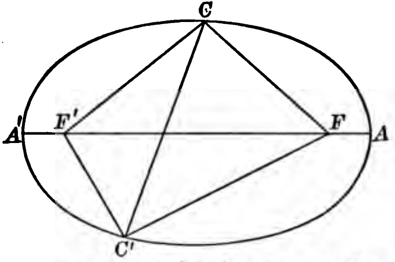


PROOF. If △ PNF is equilateral, the ⊥ PM bisects the base; hence $FM = MN =$ semi-latus rectum.

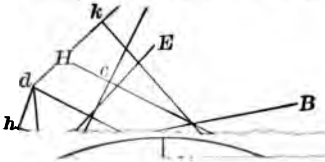
Hence $PF = FN = 2\,MN =$ latus rectum.

is BC' must pass through both P and P' (§§ 799, 808). For like reason the parabola whose focus is F and whose directrix is $B'C'$ will pass through both P and P'.

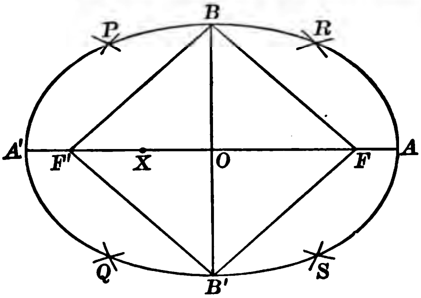
Ex. 707. Prove that the major axis is the longest chord that can be drawn in an ellipse.



Ex. 717. To draw a diameter conjugate to a given diameter in a give[illegible]



Ex. 708. If the angle $F'BF$ is a right angle, prove that $a^2 = 2b^2$.



Proof. If $F'BF = 90°$,

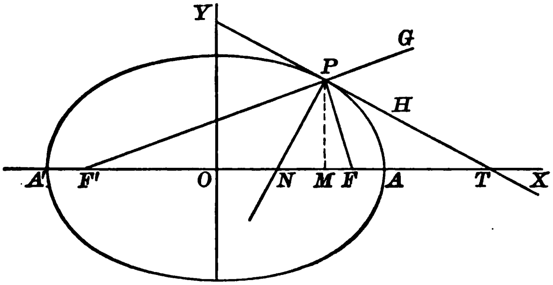
$$OM = a,$$ [illegible]

$$ON = e^2 d. \qquad (§ 874)$$

$$\therefore OM : ON = 1 : e^2 = a^2 : a^2e^2 = a^2 : c^2.$$

Ex. 709. To draw a tangent and a normal at a given point of an ellipse.

To draw a tangent to the ellipse $A'PA$ at the point P.



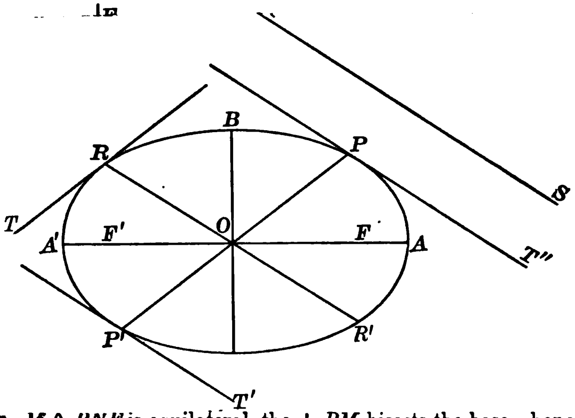
CONSTRUCTION. Draw the focal radii FP and $F'P$. Produce $F'P$ to

gent PT; then E lies in AY (§ 815),

Since $AD = AF$, $KE = EH$ (§ 187). ∴ $FK = FH$ (§ 122).

find the directrix.

CONSTRUCTION. With P as centre, and PF as radius, draw arc KH. With P' as centre, and $P'F$ as radius, draw arc DE. Draw the common tangents BC and $B'C'$; then BC and $B'C'$ will be the directrices required.



PROOF. If △ PNF is equilateral, the ⊥ PM bisects the base; hence

$$FM = MN = \text{semi-latus rectum}.$$

Hence $PF = FN = 2\,MN =$ latus rectum.

PROOF. By construction PP' is conjugate to RR'; hence PT'' and $P'T'$ are ∥ to RR' or its parallel KS (§ 877).

Ex. 711. Given the foci; it is required to describe an ellipse touching a given straight line.

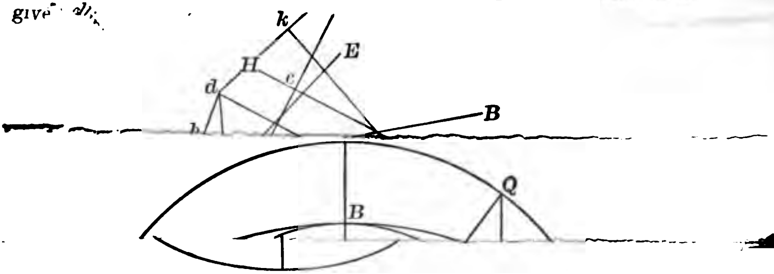
Given the foci F and F', and a tangent TP; it is required to construct the ellipse.

[illegible] $R = F'Z$.

Therefore

$$
\begin{aligned}
FR \times F'S &= F'Z \times F'S \\
&= F'A' \times F'A && (§\ 345) \\
&= (a-c)(a+c) \\
&= a^2 - c^2 \\
&= b^2. && (§\ 846)
\end{aligned}
$$

Ex. 717. To draw a diameter conjugate to a given diameter in a giv[illegible]



Ex. 712. Prove that $\overline{OF}^2 = OT \times ON$.

$$OT = \frac{a^2}{d}. \qquad (§\ 874)$$

$$ON = e^2 d. \qquad (§\ 874)$$

$$\therefore\ ON \times OT = a^2 e^2 = \overline{OF}^2.$$

Ex. 713. Prove that $OM : ON = a^2 : c^2$.

$$OM = d,$$

$$ON = e^2 d. \qquad (§\ 874)$$

$$\therefore\ OM : ON = 1 : e^2 = a^2 : a^2 e^2 = a^2 : c^2.$$

Ex. 714. The minor axis is the shortest diameter of an ellipse.

Let PP' be any diameter of the ellipse; then

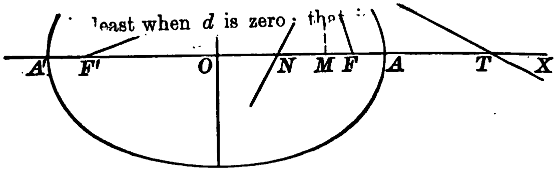
$$\overline{OP}^2 + \overline{OF}^2 = \tfrac{1}{2}(\overline{F'P}^2 + \overline{FP}^2) \qquad (\S\ 344)$$

$$= \tfrac{1}{2}(a + ed)^2 + \tfrac{1}{2}(a - ed)^2 \qquad (\S\ 854)$$

$$= a^2 + e^2d^2.$$

$$\therefore OP = \sqrt{a^2 + e^2d^2 - c^2}. \qquad (1)$$

least when d is zero; that

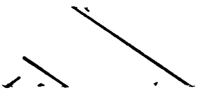


Construction. Draw the focal radii FP and $F'P$. Produce $F'P$ to

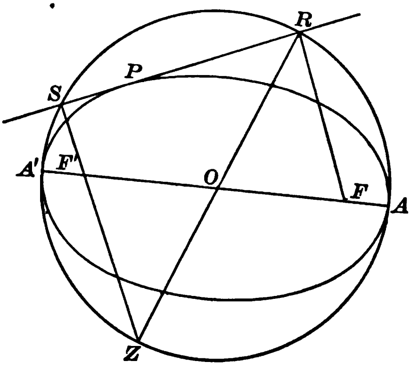
Since $AD = AF$, $KE = EH$ (§ 187). $\therefore FK = FH$ (§ 122).

find the directrix.

Construction. With P as centre, and PF as radius, draw arc KH. With P' as centre, and $P'F$ as radius, draw arc DE. Draw the common tangents BC and $B'C'$; then BC and $B'C'$ will be the directrices required.



Ex. 716. Prove that, if FR and $F'S$ are the perpendiculars dropped from the foci to any tangent, $FR \times F'S = b^2$.



Produce SF' to meet the auxiliary circle in Z, and draw RZ.

By § 867, R and S are on the auxiliary circle $ARSA'$.

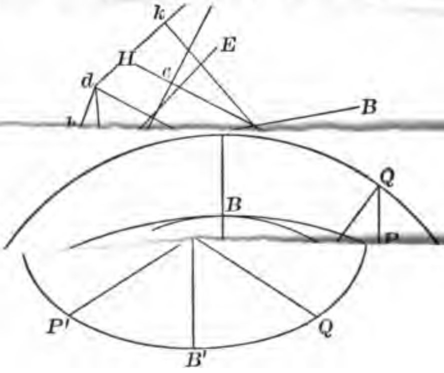
Since S is a right angle, the line RZ must be a diameter (§ 263). Hence, in $\triangle$ ORF and OZF',

$$OR = OZ,$$
$$OF = OF',$$

and $\angle FOR = \angle F'OZ$. Therefore the $\triangle$ are equal, and $FR = F'Z$.

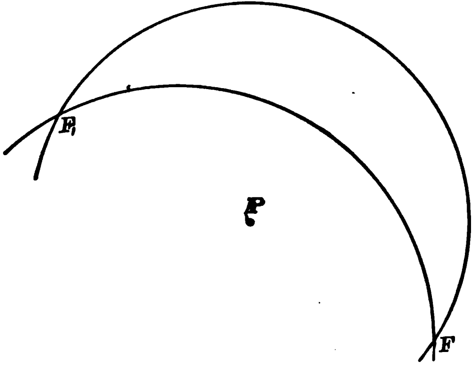
Therefore
$$\begin{aligned} FR \times F'S &= F'Z \times F'S \\ &= F'A' \times F'A \qquad (§\ 345) \\ &= (a-c)(a+c) \\ &= a^2 - c^2 \\ &= b^2. \qquad (§\ 846) \end{aligned}$$

Ex. 717. To draw a diameter conjugate to a given diameter in a give



Construction. At P draw a tangent; the diameter QOR parallel to this tangent will be conjugate to PP', by § 876.

Ex. 718. Given $2a$, $2b$, one focus, and one point on the curve, to construct the curve.

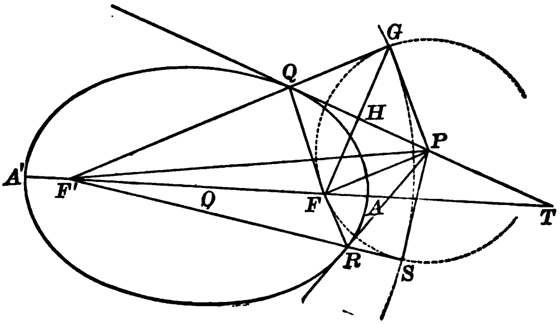


Given the focus F', the point P, $2a$, and $2b$, to construct the ellipse.

find the directrix.

CONSTRUCTION. With P as centre, and PF as radius, draw arc KH. With P' as centre, and $P'F$ as radius, draw arc DE. Draw the common tangents BC and $B'C'$; then BC and $B'C'$ will be the directrices required.

Ex. 719. If from a point P a pair of tangents PQ and PR be drawn to an ellipse, then PQ and PR will subtend equal angles at either focus.



Let $A'AQ$ be the ellipse, PQ and PR the tangents from P. To prove that $\angle PF'Q = \angle PF'R$, and $\angle PFQ = \angle PFR$.

Proof. Draw these tangents according to the method of § 868, and draw PF', PF, PG, PS.

Since $F'G = F'S = 2a$,

$PG = PS$, and $PF' = PF'$.

$\triangle PF'G$ and $PF'S$ are equal. (§ 160)

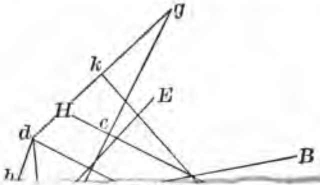
$\therefore \angle PF'Q = \angle PF'R$.

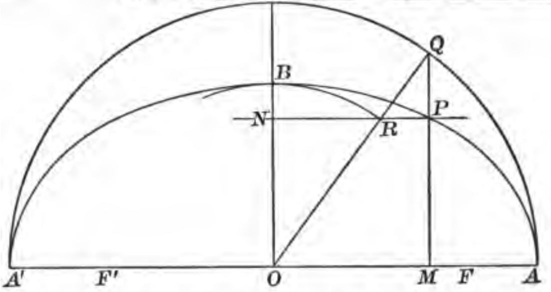
Also $\angle PGQ = \angle PSR$.

Since $PG = PF$, and $QG = QF$,

$\triangle PGQ$ and PFQ are equal. (§ 160)

Given the focus F' and the three tangents ED, BC, HI. To construct the ellipse.





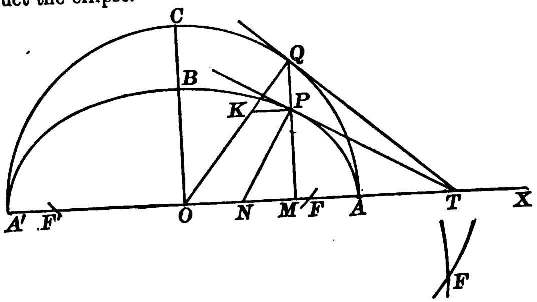
Construction. Draw the auxiliary circle AQA'. Draw $PM \perp$ to AA', and produce it to meet the auxiliary circle in Q. Join OQ, and draw $PN \parallel$ to AA', cutting OQ in R. Lay off $OB = OR$. With B as centre, and OA as radius, strike arcs cutting AA' in F and F', which

$$\therefore \square P'ORF = ab.$$

$$\therefore \square CDEF = 4 \square P'ORF = 4ab.$$

Ex. 721. To find the foci of an ellipse, having given the major axis and a straight line which touches the curve.

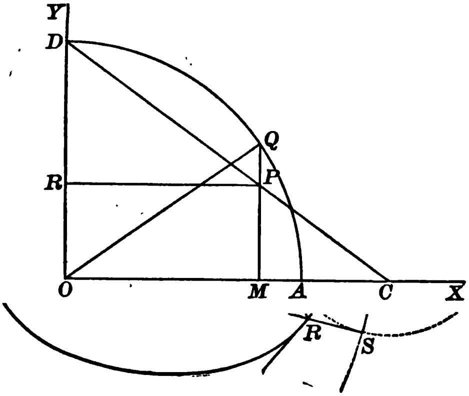
Having given the major axis AA', and the tangent PT, it is required to construct the ellipse.



Given the focus F', the point P, $2a$, and $2b$, to construct the ellipse. find the directrix.

Construction. With P as centre, and PF as radius, draw arc KH. With P' as centre, and $P'F$ as radius, draw arc DE. Draw the common tangents BC and $B'C'$; then BC and $B'C'$ will be the directrices required.

Ex. 722. A straight line moves so that its extremities are always in contact with two fixed straight lines perpendicular to each other. Prove that any point of the moving line describes an ellipse.



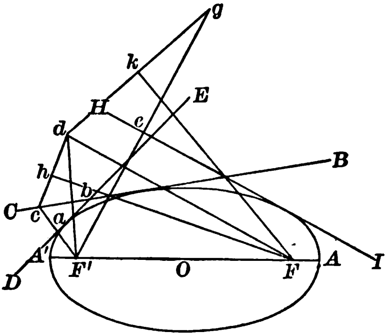
CONSTRUCTION. Through A' and F' draw the line AT. Draw $F'a \perp$ to TP, produce it, and lay off $aC = F'a$. Lay off $A'D = A'F'$. Join DC, and bisect it with the $\perp$ HF, which intersects AA' in F. Lay off $FA = F'A'$, and construct the ellipse having F' and F as foci and $A'A$ as the major axis, and it will be the curve required.

PROOF. This ellipse will have F' as a focus, A' as a vertex, and by proof in Ex. 711 will be tangent to TP at P.

Ex. 726. The area of the parallelogram formed by drawing tangents to an ellipse at the extremities of any pair of conjugate diameters is

Ex. 723. To construct an ellipse, having given one of the foci and three tangents.

Given the focus F' and the three tangents ED, BC, HI. To construct the ellipse.



PROOF. Let the tangent at R meet OA produced in T; join $P'T$. Draw the ordinates PM, RN, and $P'M'$; then, since RT is ∥ to $P'O$,

$$\square P'ORF = 2 \times \triangle P'OT = OT \times M'P'. \qquad (\S\S\ 365, 368)$$

Since $MP = M'P'$, we have

$$M'P' : ON = b : a. \qquad (\S\ 879)$$

$$\therefore OT \times M'P' : OT \times ON = b : a = ab : a^2.$$

But

$$OT \times ON = a^2. \qquad (\S\ 871)$$

$$\therefore OT \times M'P' = ab.$$

$$\therefore \square P'ORF = ab.$$

$$\therefore \square CDEF = 4 \square P'ORF = 4ab.$$

ADVERTISEMENTS.

MATHEMATICS.

WENTWORTH'S SERIES.

THE extraordinary success of Professor Wentworth's mathematical text-books must have a cause: the cause assigned by those in the best position to judge is that the books are *just right.* The author has aimed for the center of the shield and has struck it.

Two words indicate the two leading characteristics of these remarkable books: scholarship and text-book availability. They combine mathematical brilliancy with all the elements of schoolroom popularity. The authors aim was to be helpful. He has made a study of his subjects with the psychology and probable capacity of the students constantly in mind. His books enable the average student not merely to "learn something," but to MASTER the study, and at the same time they give the brighter ones plenty to exercise their faculties. There is no attempt to make a parade of learning. On the other hand, there is no absolution from solid work. Everything is made simple, practical, direct, and thorough. Mental energy is economized. The teacher has to use his strength only in the necessary work of teaching, and the pupil his only in the necessary work of acquiring.

To learn by doing, and to learn one step thoroughly before the next is attempted, are the chief elements of the method, and yet no books give pupil or teacher less of the tread-mill feeling. The consciousness of mastery constantly cheers and invigorates the student, while the teacher has the satisfaction of wielding an instrument fitted to the hand. All will recognize in these characteristics the marks of ideal text-books.

At all points one finds not only the proficient mathematician, but the practical teacher. Where other books leave teacher and pupils to struggle out of difficulties for themselves, one finds evidence of the most careful study and most thorough work. These are some of the characteristics that have made these books both taking at first sight and satisfactory in use.

Arithmetics for Common Schools.

For other details, see pages 184–186.

Wentworth's Mental Arithmetic.

Mailing price, 35 cents; for introduction, 30 cents.

Wentworth's Elementary Arithmetic.

Mailing price, 40 cents; for introduction, 30 cents.

Wentworth's Practical Arithmetic. *[Nearly ready.*

Wentworth and Reed's First Steps in Number.

For prices of this book, see page 189.

Wentworth's Primary Arithmetic.

Mailing price, 40 cents; for introduction, 30 cents.

Wentworth's Grammar School Arithmetic.

**Mailing price, 75 cents; for introduction 65 cents.
For full description of Wentworth's Common School Arithmetics, see Common School Catalogue.**

A High School Arithmetic.

**By G. A. Wentworth, recently Prof. of Mathematics, Phillips Exeter Academy, and the late Dr. Thomas Hill, Portland, Me. 12mo. Roan back. 367 pages. Mailing price, $1.10; for introduction, $1.00. Answers free, on teachers' order.
Same. Abridged Edition. For Grammar Schools. 288 pages (*including Answers*). Mailing price, $1.00; for introduction, 75 cents.**

THIS book is intended for high and normal schools and academies, It is particularly adapted to general mental discipline, to preparation for higher studies, business, or professional life.

Wentworth's First Steps in Algebra.

By G. A. Wentworth, recently Professor of Mathematics in Phillips Exeter Academy. 12mo. Half leather. vii + 184 pages. Mailing price, 70 cents; for introduction, 60 cents.

THIS book is specially prepared for pupils in the upper grades of grammar schools and high schools.

A. Newton Ebaugh, *Professor of Mathematics, Polytechnic Institute, Baltimore, Md.:* It is well adapted to the work of the first year and is sufficiently comprehensive to give all the essentials and at the same time not cloud and bewilder the student with non-essentials.

W. B. Sutliff, *Professor of Mathematics, State Normal School, Bloomsburg, Pa.:* It is a very carefully prepared work and will meet with hearty approval by all teachers of the grade for which it is intended.

Wentworth's School Algebra.

By G. A. WENTWORTH, recently Professor of Mathematics, Phillips Exeter Academy. Half morocco. v + 362 pages. Mailing price, $1.25; for introduction, $1.12. *Answers in pamphlet form, free, on teachers' orders.*

THE School Algebra is offered as exactly right for the usual high school and academic courses. It gives a thorough and practical treatment of the principles of Algebra up to and including the binomial theorem, and is strictly in line throughout with the author's College Algebra. For college preparation it is particularly well suited. The problems in this book are nearly all new, either original or selected from recent examination papers, and are graded with the utmost care.

Theodore L. Sewall, *Principal of Girls' Classical School, Indianapolis:* An admirable book, like all of the Wentworth Series.

George Gilbert, *Principal of Chester Academy, Pa.:* The best book for its grade yet issued.

George E. Gay, *Principal of High School, Malden, Mass.:* Better adapted to use in high schools than any other.

T. W. Palmer, *Prof. of Mathematics, University of Alabama:* An admirable work.

C. D. Schmitt, *University of Tennessee:* For the work intended, I do not think it can be surpassed.

L. R. Hunt, *Principal of High School, Corning, N.Y.:* It meets my desires completely.

W. P. Durfee, *Prof. of Mathematics, Hobart College, N.Y.:* An admirable book for college preparation.

J. S. Slocum, *Principal South Division High School, Chicago:* I have used it in connection with the preparation of a class for college and have been pleased with its clear definitions, logical arrangement, and happy selection of both examples and problems.

George W. Price, *Teacher of Mathematics, High School, Council Bluffs, Ia.:* Without doubt the best book out for high schools.

Erastus Test, *Principal Purdue University Prep. Dept.:* I regard it as the *ne plus ultra* in the line of a school Algebra.

Frank E. Thompson, *Principal High School, Newport, R.I.:* A thorough knowledge of the contents of the book will enable a pupil to pass an entrance examination to any college.

David Eugene Smith, *Professor of Mathematics, State Normal and Training School, Ypsilanti Mich.:* I have examined it with a good deal of care. It seems well adapted to the needs of our schools — even better adapted than the author's former work, which I have used and recommended. The improvements to be found in this work are such as will meet the approval of all teachers.

O. S. Westcott, *Principal North Division High School, Chicago:* The student who finishes it will be splendidly prepared to grapple with the beautiful discussions of higher Algebra.

Wentworth's Higher Algebra.

By G. A. Wentworth, recently Prof. of Mathematics, Phillips Exeter Academy. Half morocco. 528 pages. Mailing price, $1.55; for introduction, $1.40. *Answers in pamphlet form, free, on teachers' orders.*

THIS work is designed to prepare thoroughly for colleges and scientific schools, and to furnish in addition what is needed for the *general student* in such institutions. It provides *in a single book a complete course parallel* to the course provided by the School and College Algebras together.

Preparatory schools and academies of high grade, especially such as give the pupils a thorough preliminary drill in arithmetic, normal schools, seminaries, and many colleges, will find it specially suited to their requirements.

Davis Garber, *Professor of Mathematics and Astronomy, Muhlenberg College, Allentown, Pa.:* It meets the wants of a general college course better than any of the author's other works on the same subject.

David Eugene Smith, *Professor of Mathematics, State Normal School, Ypsilanti, Mich.:* As a one-volume work, intended for advanced students, I do not think that it has any superior. It embodies all of the subjects of general algebra that are really necessary for higher work.

Wentworth's College Algebra.

By G. A. Wentworth, recently Prof. of Mathematics, Phillips Exeter Academy. Half morocco. 500 pages. Mailing price, $1.65; for introduction, $1.50. *Answers in pamphlet form, free, on teachers' orders.*

THIS is what its name indicates, a text-book for colleges and scientific schools. The first part is simply a concise review of the principles of Algebra preceding quadratics, with enough examples to illustrate and enforce the principles. Room is thus left for a full discussion of the higher topics. The endeavor has been to give in matter and methods the best training in algebraic analysis at present attainable. The work covers a full year, but by omitting starred sections and problems, the instructor can arrange a half-year course.

William Beebe, *Assistant Prof. of Mathematics and Astronomy, Yale College:* I find it characterized by the clearness and method of all Professor Wentworth's books, and am particularly struck with the amount of matter in the Algebra.

Dascom Greene, *Emeritus Prof. of Mathematics, and Astronomy, Rensselaer Polytechnic Institute, Troy, N.Y.:* To the student well grounded in the elements it furnishes an excellent practical course in the higher branches of the subject.

Wentworth's Elements of Algebra.

By G. A. WENTWORTH, recently Professor of Mathematics, Phillips Exeter Academy, and author of *Geometry*, *Trigonometry*, etc. Half morocco. x + 325 pages. Mailing price, $1.25; for introduction, $1.12. *Answers bound separately in pamphlet form.*

THIS book is designed for high schools and academies, and contains an ample amount for admission to any college. In grade and amount it is exactly equivalent to the *School Algebra*, and as the problems are entirely different, it may be used as an alternative when a change of problems is found desirable.

Wentworth's Complete Algebra.

Includes the subjects usually taught in Colleges. Half morocco. 525 pages. Mailing price, $1.55; for introduction, $1.40. *Answers bound separately in pamphlet form.*

THIS work consists of the author's Elementary Algebra, with about one hundred and eighty-five pages additional.

Wentworth's Shorter Course in Algebra.

Half morocco. 258 pages. By mail, $1.10; for introduction, $1.00. *Answers in pamphlet form, free, on teachers' orders.*

THIS book was prepared and is recommended only for a special use. It was based upon the author's Elements of Algebra, and contains all the essential subjects treated in that book, but with fewer examples, so as to make a one-year's course.

Algebraic Analysis.

By G. A. WENTWORTH, A.M., recently Prof. of Mathematics, Phillips Exeter Academy; J. A. MCLELLAN, LL.D., Inspector of Normal Schools, and Conductor of Teachers' Institutes for Ontario, Canada; and J. C. GLASHAN, Inspector of Public Schools, Ottawa, Canada. Part I. concluding with Determinants. 12mo. Half leather. x + 418 pages. By mail, $1.60; to teachers and for introduction, $1.50.

THIS work is intended to supply students of Mathematics with a well-filled storehouse of solved examples and unsolved exercises in the application of the fundamental theorems and processes of pure Algebra.

Wentworth's New Plane Geometry.

By G. A. Wentworth, recently Professor of Mathematics, Phillips Exeter Academy. 12mo. x + 242 pages. Mailing price, 85 cents; for introduction, 75 cents.

Wentworth's New Plane and Solid Geometry.

By G. A. Wentworth. 12mo. Half morocco. xi + 386 pages. Mailing price, $1.40; for introduction, $1.25. The book now includes a treatise on Conic Sections (Book IX.).

ALL the distinguishing characteristics of the first edition have been retained. The subject is treated as a branch of practical logic, the object of which is to detect and state with precision the successive steps from premise to conclusion.

In each proposition a concise statement of what is given is printed in one kind of type, of what is required in another, and the demonstration in still another. The reason for each step is indicated in small type between that step and the one following; and the author thus avoids the necessity of interrupting the process of demonstration to cite a previous proposition. The number of the section on which the reason depends is, however, placed at the side of the page; and the pupil should be prepared, when called upon, to give the proof of each reason. Each distinct assertion in the demonstrations and each particular direction in the construction of the figures begins a new line, and in no case is it necessary to turn the page in reading a demonstration.

In the new edition will be found a few changes in the order of the subject-matter. Some of the demonstrations have been given in a more concise and simple form. The diagrams, with which especial care was taken originally, have been re-engraved and materially improved. The shading, which has been added to many of the figures, has proved a great help to the constructive imagination of pupils. The theory of limits — the value of which the author emphasizes — has been presented in the simplest possible way, and its application made easy of comprehension.

But the great feature of this edition is the introduction of nearly seven hundred original exercises, consisting of theorems, problems of construction, and problems of computation, carefully graded and adapted to beginners in Geometry.

W. A. Moody, *Prof. of Mathematics, Bowdoin College:* I consider it an improvement on the old edition, noticeably in the better figures and general finer appearance of the book and in the numerous "original" exercises.

George W. Sawin, *late Instructor in Mathematics in Harvard College:* I have always regarded the old edition of Wentworth's Geometry, as, on the whole, the best text-book of its kind in English. The new edition is, in my opinion, a far better book than the old. It may interest you to learn that I recently found that *out of a class of one hundred and ten in Solid Geometry, one hundred and one had been prepared in Wentworth's Plane Geometry.*

J. R. French, *Prof. of Mathematics, Syracuse University:* It seems to be a great improvement upon the former edition. We shall doubtless continue using it.

W. C. Bartol, *Professor of Mathematics, Bucknell University:* I believe it is a decided improvement on the old edition, and shall use it in my classes.

Lyman Hall, *Prof. of Mathematics, Georgia School of Technology:* I find the new edition of Wentworth's Geometry admirably adapted to our needs here. I think the author has accomplished the great end of giving clear and concise proofs in the minimum of space, and has materially improved, without rendering too difficult, the original exercises and numerical examples.

W. Hoover, *Prof. of Mathematics, Ohio University:* I have introduced into my classes the new edition of Wentworth's Geometry, and find it very much of an improvement upon the old. . . . I never had such good results in my geometry classes.

E. Miller, *Prof. of Mathematics, University of Kansas:* The book is a very superior one, and grows in our estimation because it is exactly suited to our needs. It is a book that compels a student to think and invent new methods and demonstrations.

J. M. Taylor, *Prof. of Mathematics, University of Washington, Seattle:* The *best* book on geometry has been made *better* by the revision.

Fred T. Wright, *Principal of High School, Jackson, Mich.:* I know of no other text-book in Geometry that I would wish to substitute for it, simply because I know of no book which is its equal.

Ella I. Harris, *Teacher of Mathematics, St. Mary's Hall, Indianapolis:* For the past four years I have been using it in my classes, and find it for clearness and simplicity, and for original work offered, superior to any class book it has been my privilege to examine.

F. L. Morse, *Professor of Mathematics, Hanover College, Hanover, Ind.:* There is no text-book on the subject that has given me as much satisfaction. It stands the test of the recitation room.

W. N. Ferris, *Principal of Industrial School, Big Rapids, Mich.:* For more than one year I have been using it. The work is unexcelled. It has more than met our most sanguine expectations and my students invariably pronounce it the best geometry.

J. Fred Smith, *Principal of Academy, Iowa College, Grinnell, Iowa:* It has long been the text-book in the Academy. We have especially prized it for accuracy of statement, clearness with brevity, and for the important place given to original work on the part of the student.

Wentworth's Trigonometries.

By G. A. Wentworth.

Plane and Solid Geometry, and Plane Trigonometry.

12mo. Half morocco. 490 pages. Mailing price, $1.55; for introduction, $1.40.

New Plane Trigonometry.

12mo. Paper. 134 pages. Mailing price, 45 cents; for introduction, 40 cents. The old edition is still issued.

New Plane Trigonometry, with Tables.

8vo. Cloth. 249 pages. Mailing price, $1.00; for introduction, 90 cents. The old edition is still issued.

New Plane and Spherical Trigonometry.

12mo. Half morocco. 214 pages. Mailing price, 95 cents; for introduction, 85 cents. The old edition is still issued.

New Plane and Spherical Trigonometry, with Tables.

8vo. Half morocco. 315 pages. Mailing price, $1.30; for introduction, $1.20. The old edition is still issued.

New Plane Trigonometry, and Surveying, with Tables.

8vo. Half morocco. 305 pages. Mailing price, $1.30; for introduction, $1.20.

New Plane and Spherical Trigonometry and Surveying, with Tables.

8vo. Half morocco. 368 pages. Mailing price, $1.50; for introduction, $1.35.

New Plane and Spherical Trigonometry, Surveying, and Navigation.

12mo. Half morocco. 412 pages. Mailing price, $1.30; for introduction, $1.20.

THE aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view to awaken a real love for the study. Much time and labor have been spent in devising the simplest proofs for the propositions, and in exhibiting the best methods of arranging the logarithmic work. Answers are included.

The New Plane Trigonometry gives sufficient practice in the *radian* as the unit of angular measure, in solving simple trigonometric equations, in solving right triangles without the use of logarithms, and in solving problems in goniometry.

It also contains the latest entrance examination papers of some of the leading colleges and scientific schools; and a chapter on the development of functions of angles in infinite series.